

Frames arising from irreducible solvable actions Part I

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Abstract

Let G be a simply connected, connected completely solvable Lie group with Lie algebra $\mathfrak{g} = \mathfrak{p} + \mathfrak{m}$. Next, let π be an infinite-dimensional unitary irreducible representation of G obtained by inducing a character from a closed normal subgroup $P = \exp \mathfrak{p}$ of G . Additionally, we assume that $G = P \rtimes M$, $M = \exp \mathfrak{m}$ is a closed subgroup of G , $d\mu_M$ is a fixed Haar measure on the solvable Lie group M and there exists a linear functional $\lambda \in \mathfrak{p}^*$ such that the representation $\pi = \pi_\lambda = \text{ind}_P^G(\chi_\lambda)$ is realized as acting in $L^2(M, d\mu_M)$. Making no assumption on the integrability of π_λ , we describe explicitly a discrete subgroup $\Gamma \subset G$ and a vector $\mathbf{f} \in L^2(M, d\mu_M)$ such that $\pi_\lambda(\Gamma)\mathbf{f}$ is a tight frame for $L^2(M, d\mu_M)$. We also construct compactly supported smooth functions \mathbf{s} and discrete subsets $\Gamma \subset G$ such that $\pi_\lambda(\Gamma)\mathbf{s}$ is a frame for $L^2(M, d\mu_M)$.

1 Introduction and preliminaries

Let G be a locally compact group, and let π be a strongly continuous unitary irreducible representation of G acting in an infinite-dimensional Hilbert space \mathcal{H}_π . Next, let Γ be a discrete subset of G and fix $\mathbf{f} \in \mathcal{H}_\pi$. We say that $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ is a **frame** for \mathcal{H}_π if there exist positive constants $a \leq b$ (frame bounds) such that

$$a \|\mathbf{h}\|_{\mathcal{H}_\pi}^2 \leq \sum_{\gamma \in \Gamma} |\langle \mathbf{h}, \pi(\gamma)\mathbf{f} \rangle_{\mathcal{H}_\pi}|^2 \leq b \|\mathbf{h}\|_{\mathcal{H}_\pi}^2$$

for any vector \mathbf{h} in \mathcal{H}_π . The frame operator S is defined as

$$S\mathbf{h} = \sum_{\gamma \in \Gamma} \langle \mathbf{h}, \pi(\gamma)\mathbf{f} \rangle_{\mathcal{H}_\pi} \pi(\gamma)\mathbf{f} \quad (\mathbf{h} \in \mathcal{H}_\pi)$$

and if $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ is a frame for \mathcal{H}_π then S is invertible and every vector \mathbf{h} in \mathcal{H}_π admits the expansion

$$\mathbf{h} = \sum_{\gamma \in \Gamma} \langle \mathbf{h}, S^{-1}\pi(\gamma)\mathbf{f} \rangle_{\mathcal{H}_\pi} \pi(\gamma)\mathbf{f}$$

with convergence in the norm of \mathcal{H}_π . If $a = b$ then $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ is called a **tight frame** and every vector $\mathbf{h} \in \mathcal{H}_\pi$ admits the simpler series expansion

$$\mathbf{h} = \sum_{\gamma \in \Gamma} \left\langle \mathbf{h}, \pi(\gamma) \frac{\mathbf{f}}{\sqrt{a}} \right\rangle_{\mathcal{H}_\pi} \pi(\gamma) \frac{\mathbf{f}}{\sqrt{a}}.$$

We are interested in finding conditions under which there exist a discrete subset $\Gamma \subset G$ and a vector $\mathbf{f} \in \mathcal{H}_\pi$ such that the collection $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ is a (tight) frame for \mathcal{H}_π . Moreover, if there exist frames generated by π , we would also like to present an explicit procedure for the construction of Γ and \mathbf{f} such that $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ is a frame for \mathcal{H}_π . If π is an irreducible and integrable representation, then the coorbit theory [16, 11, 12] which was developed by Feichtinger and Gröchenig provides a powerful and flexible discretization scheme. For example, the theory of coorbit has proved to be quite successful in the context of shearlets [5, 6]. In a more general direction, the work contained in [15, 16] addresses the case where $G = \mathbb{R}^d \rtimes H$ (where H is a closed subgroup of $\text{GL}(\mathbb{R}^d)$) is the semi-direct product group with multiplication law given by

$$(x, M)(x', M') = (x + Mx', MM')$$

and π is the quasiregular representation of G which is realized as acting in $L^2(\mathbb{R}^d)$ such that

$$[\pi(x, id_H)\mathbf{f}](t) = \mathbf{f}(t - x) \text{ and } [\pi(0, M)\mathbf{f}](t) = |\det M|^{-1/2} \mathbf{f}(M^{-1}t). \quad (1.1)$$

It is well-known that if the quasiregular representation π is irreducible then it is integrable as well [16]. As such, the coorbit machinery of Feichtinger and Gröchenig can be applied to construct frames for a large class of Banach spaces. In fact, it is proved in [19] that if $\mathbf{f} \in L^2(\mathbb{R}^d)$ satisfies some decay, smoothness and vanishing moments conditions, then there exists a discrete subset $\Gamma \subset \mathbb{R}^d \rtimes H$ such that $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ is a wavelet frame for the Hilbert space on which the representation π is acting [19]. Unfortunately, the theory of coorbit heavily depends on the integrability of π . Thus, a large class of irreducible representations are automatically excluded because they do not fit within the scope of the theory.

The main objective of the present work is to provide a method for constructing frames arising from the action of irreducible representations of some solvable Lie groups which does not depend on any type of integrability condition on the representations of interest. To be more specific, let $G = P \rtimes M$ be a simply connected, connected completely solvable Lie group such that $P = \exp \mathfrak{p}$ and $M = \exp \mathfrak{m}$ are closed solvable Lie subgroups of G . Moreover, we shall assume that there exists a unitary character χ of P such that

$$\pi = \text{ind}_P^G(\chi)$$

is an irreducible representation of G realized as acting on functions defined on the conormal subgroup M . The space on which the representation π is acting consists of measurable functions on M which are square-integrable with respect to a fixed Haar measure on M . Without imposing any additional assumption on the integrability of the representation π , we shall provide a unified and explicit procedure which can be exploited to construct discrete tight frames, and smooth compactly supported frames generated by the action of π .

1.1 Notation

- Let T be a linear operator acting on a vector space spanned by an ordered basis \mathfrak{B} . The matrix representation of the linear operator T is denoted $[T]_{\mathfrak{B}}$ and T^* stands for the adjoint of the linear operator T .

- The transpose of a matrix M is denoted M^T . Moreover, the inverse transpose of M is written as M^\top .
- Let Q be a linear operator acting on an n -dimensional real vector space V . The norm of the matrix Q induced by the max-norm of the vector space V is given by

$$\|Q\|_\infty = \sup \{ \|Qv\|_{\max} : v \in V \text{ and } \|v\|_{\max} = 1 \}$$

and the max-norm of an arbitrary vector is

$$\|v\|_{\max} = \max \{ |v_k| : 1 \leq k \leq n \}.$$

- Let A be a Lebesgue measurable subset of \mathbb{R}^d . The Lebesgue measure of A is denoted $|A|$.
- Let \mathfrak{s} be Lie algebra, and let λ be a linear functional in the dual of \mathfrak{s} which we denote by \mathfrak{s}^* . A subalgebra (or ideal) \mathfrak{p} of \mathfrak{s} is said to be subordinated to the linear functional λ if

$$[\mathfrak{p}, \mathfrak{p}] = \mathbb{R}\text{-span} \{ [X, Y] : X, Y \in \mathfrak{p} \}$$

is contained in the kernel of the linear functional λ .

1.2 Completely solvable Lie groups

Let \mathfrak{g} be a finite-dimensional Lie algebra over \mathbb{R} . Given subsets $\mathfrak{r}, \mathfrak{s} \subseteq \mathfrak{g}$, we define $[\mathfrak{r}, \mathfrak{s}]$ as the linear span of vectors

$$[X, Y] = XY - YX$$

such that $X \in \mathfrak{r}$ and $Y \in \mathfrak{s}$. Put $\mathfrak{g}_{(0)} = \mathfrak{g}$ and define in a recursive fashion

$$\mathfrak{g}_{(k)} = [\mathfrak{g}_{(k-1)}, \mathfrak{g}_{(k-1)}].$$

The sequence

$$\mathfrak{g}_{(0)} \supseteq \mathfrak{g}_{(1)} \supseteq \mathfrak{g}_{(2)} \cdots$$

is called the **derived series** of the Lie algebra \mathfrak{g} . In a similar fashion, we define the **descending central series** of \mathfrak{g} inductively as follows

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}].$$

A Lie algebra is **solvable** if there exists a natural number n such that $\dim \mathfrak{g}_{(n)} = 0$. Additionally, a Lie algebra \mathfrak{g} is called **nilpotent** if $\dim \mathfrak{g}^{(n)} = 0$ for some natural number n . From the definitions provided above, it is clear that $[\mathfrak{g}_{(k-1)}, \mathfrak{g}_{(k-1)}] \subseteq [\mathfrak{g}^{(k-1)}, \mathfrak{g}]$ for all k . As such, every nilpotent Lie algebra is necessarily solvable. However, there exist solvable Lie algebras which are not nilpotent. Additionally, a solvable Lie algebra \mathfrak{g} is called **completely solvable** if for any $Z \in \mathfrak{g}$, the spectrum of the linear operator

$$\mathfrak{g} \ni X \mapsto [Z, X] = ZX - XZ$$

is a subset of the reals. If \mathfrak{g} is nilpotent then for any given $Z \in \mathfrak{g}$, the spectrum of the linear operator $X \mapsto [Z, X]$ must coincide with $\{0\}$. Furthermore, it is well-known that if \mathfrak{g} is a completely solvable Lie algebra then there exists an ordered basis $\mathfrak{B} = (Z_1, \dots, Z_d)$ of \mathfrak{g} such that each $\mathfrak{g}_i = \mathbb{R}\text{-span} \{Z_1, \dots, Z_i\}$ is an ideal in \mathfrak{g} . Such a basis \mathfrak{B} is called a **strong Malcev basis** for the Lie algebra. If G is a simply connected, connected Lie group with a completely solvable finite-dimensional real Lie algebra \mathfrak{g} then, G is called a **completely solvable** Lie group. As it is well-known, if G is completely solvable then the exponential map defines a bi-analytic bijection between \mathfrak{g} and G , and the inverse of the exponential map is denoted \log . If $\mathfrak{B} = (Z_1, \dots, Z_d)$ is a strong Malcev basis for \mathfrak{g} then the map

$$(t_1, \dots, t_d) \mapsto \exp(t_1 Z_1) \exp(t_2 Z_2) \cdots \exp(t_d Z_d)$$

defines an analytic diffeomorphism from \mathbb{R}^d onto G (see [39], Theorem 3.18.11.) This map induces a system of coordinates on G called the **canonical coordinates of the second kind**. Let \mathfrak{p} be a subalgebra of \mathfrak{g} . Next, let $P = \exp \mathfrak{p}$. If \mathfrak{p} is subordinated to the linear functional λ , then

$$\chi_\lambda(\exp X) = e^{2\pi i \langle \lambda, X \rangle} = e^{2\pi i \lambda(X)}$$

defines a continuous one-dimensional representation of P , and χ_λ is called a **unitary character** of P . Given an element $x = \exp(X) \in G$, we define the linear maps $\text{ad}(X)$, $\text{Ad}(x)$ acting on the Lie algebra of G as follows: $\text{ad}(X)(Y) = [X, Y]$ and $\text{Ad}(x) = e^{\text{ad}(X)}$. Let

$$\mathbf{O}_\lambda = \{ \text{Ad}(x^{-1})^* \lambda : x \in G \}$$

be the **coadjoint orbit** of the linear functional λ , and let \mathfrak{p}^\perp be the orthogonal complement of \mathfrak{p} in the dual vector space \mathfrak{g}^* . That is,

$$\mathfrak{p}^\perp = \{ \ell \in \mathfrak{g}^* : \ell(X) = 0 \text{ for all } X \in \mathfrak{p} \}.$$

It is well-known that the induced representation

$$\pi_\lambda = \text{ind}_P^G(\chi_\lambda)$$

is irreducible if and only if [38]

1. $\dim(\mathfrak{p}) = \dim(\mathfrak{g}) - \frac{1}{2} \dim(\mathbf{O}_\lambda)$
2. Pukansky's condition holds. That is, $\lambda + \mathfrak{p}^\perp \subseteq \mathbf{O}_\lambda$.

Moreover, for every irreducible unitary representation of G , there exists a linear functional $\ell \in \mathfrak{g}^*$ and a subalgebra \mathfrak{p}_ℓ subordinated to ℓ such that the given representation is unitarily equivalent to the induced representation $\pi_\ell = \text{ind}_{\exp \mathfrak{p}_\ell}^G(\chi_\ell)$. Let us now suppose that G is a solvable Lie group satisfying the following.

Condition 1 *G is connected, simply connected completely solvable Lie group of the type $G = P \rtimes M$ where $P = \exp \mathfrak{p}$, $M = \exp \mathfrak{m}$ are closed subgroups of G . Moreover, there exists a linear functional λ in \mathfrak{p}^* such that the induced representation $\pi_\lambda = \text{ind}_P^G(\chi_\lambda)$ which is realized as acting in $L^2(M, d\mu_M)$ is an irreducible representation of G ($d\mu_M$ is a fixed Haar measure on the solvable subgroup M .)*

Let $(A_1, A_2, \dots, A_{n_2})$ be a fixed strong Malcev basis for the Lie algebra \mathfrak{m} . For a fixed $m \in M$, there exists a unique element $a = (a_1, a_2, \dots, a_{n_2}) \in \mathbb{R}^{n_2}$ such that

$$m = \exp(a_1 A_1) \cdots \exp(a_{n_2} A_{n_2}).$$

Letting dA be the Lebesgue measure on the Lie algebra of M , a left Haar measure on M is up to multiplication by a constant uniquely determined as follows (see [10] Page 90)

$$d\mu_M \left(\exp \left(\sum_{k=1}^{n_2} a_k A_k \right) \right) = d\mu_M (\exp(A)) = \left| \det \left(\frac{id - e^{-\text{ad}(A)}}{\text{ad}(A)} \right) \right| dA.$$

Put

$$w(A) = \left| \det \left(\frac{id - e^{-\text{ad}(A)}}{\text{ad}(A)} \right) \right|.$$

The function w is a non-vanishing smooth positive function on the Lie algebra \mathfrak{m} satisfying $w(0) = 1$. Moreover, the modular function of M is given by

$$\Delta_M (\exp(a_1 A_1) \cdots \exp(a_{n_2} A_{n_2})) = |\det \text{Ad} (\exp(a_1 A_1) \cdots \exp(a_{n_2} A_{n_2}))|^{-1}$$

and if \mathfrak{m} is a nilpotent algebra then $\text{ad}(A)$ is nilpotent and $w(A) = 1$. Put $\dim(P) = n_1$ and $\dim(M) = n_2$. The mapping

$$(x, a) = (x_1, x_2, \dots, x_{n_1}, a_1, a_2, \dots, a_{n_2}) \mapsto \exp \left(\sum_{k=1}^{n_1} x_k X_k \right) \exp(a_1 A_1) \cdots \exp(a_{n_2} A_{n_2})$$

defines an analytic diffeomorphism between the Lie algebra \mathfrak{g} and its Lie group G . This diffeomorphism induces a system of coordinates on the Lie group G . Since every element $g \in G$ is uniquely written as

$$g = (p, n) \in P \times M$$

it follows that

$$[\pi_\lambda(g) \mathbf{f}](m) = [\pi_\lambda(p, n) \mathbf{f}](m) = e^{2\pi i \langle \lambda, \log(m^{-1}pm) \rangle} \mathbf{f}(n^{-1}m).$$

If $p = \exp(X)$ for some $X = \sum_{k=1}^{n_1} x_k X_k \in \mathfrak{p}$ then there exist some real numbers a_k such that

$$[\pi_\lambda(p, n) \mathbf{f}](m) = e^{2\pi i \langle \lambda, e^{-\text{ad}(a_{n_2} A_{n_2})} \dots e^{-\text{ad}(a_2 A_2)} e^{-\text{ad}(a_1 A_1)} X \rangle} \mathbf{f}(n^{-1}m).$$

Next, define the linear map $\mathbf{C}(a) : \mathfrak{p} \rightarrow \mathfrak{p}$ such that

$$\mathbf{C}(a) = \mathbf{C}(a_1, a_2, \dots, a_{n_2}) = \left(e^{-\text{ad}(a_{n_2} A_{n_2})} \dots e^{-\text{ad}(a_2 A_2)} e^{-\text{ad}(a_1 A_1)} \right) \Big|_{\mathfrak{p}}. \quad (1.2)$$

Given $r = (r_1, \dots, r_{n_2}) \in \mathbb{R}^{n_2}$, let

$$\mathbf{Q}(a, \lambda, X) = \langle \mathbf{C}(a)^* \lambda, X \rangle$$

and

$$\mathbf{e}(r) = \exp(r_1 A_1) \cdots \exp(r_{n_2} A_{n_2}).$$

Then

$$\left[\pi_\lambda \left(\exp \left(\sum_{k=1}^{n_1} x_k X_k \right) \mathbf{e}(t) \right) \mathbf{f} \right] (\mathbf{e}(a)) = e^{2\pi i \langle \mathbf{Q}(a, \lambda, x) \rangle} \mathbf{f}(\mathbf{e}(t)^{-1} \mathbf{e}(a)). \quad (1.3)$$

1.3 Wavelet theory and time-frequency analysis

In order to convince the reader that the class of groups under investigation is relevant to wavelet theory and time-frequency analysis experts, we shall present a few examples belonging to the class of groups under consideration.

- **(Affine group)** Let G be the $\mathbf{ax}+\mathbf{b}$ group with Lie algebra \mathfrak{g} spanned by X_1, A_1 such that

$$[A_1, X_1] = X_1.$$

Given a linear functional $\lambda = \lambda_1 X_1^*$ such that λ_1 is a non-zero real number, π_λ is realized as acting on the Hilbert space $L^2(M, d\mu_M)$ as follows. For a square-integrable function \mathbf{f} with respect to the fixed Haar measure $d\mu_M$,

$$[\pi_\lambda(\exp(xX_1)\exp(tA_1))\mathbf{f}](\exp(aA_1)) = e^{2\pi i x e^{-a}\lambda_1} \mathbf{f}(\exp((a-t)A_1)).$$

Since G is isomorphic to $\mathbb{R} \rtimes e^{\mathbb{R}}$ with multiplication law

$$(x, e^t)(y, e^s) = (x + e^t y, e^{t+s}),$$

we may in fact remodel the representation π_λ as acting on $L^2((0, \infty), \frac{dh}{h})$ such that

$$[\pi_\lambda(\exp(xX_1)\exp(tA_1))\mathbf{f}](h) = e^{\frac{2\pi i x \lambda_1}{h}} \mathbf{f}\left(\frac{h}{e^t}\right).$$

- **(Toeplitz shearlet groups)** Let us consider the Lie algebra spanned by

$$X_1, X_2, \dots, X_{n_1}, A_1, \dots, A_{n_2-1}, A_{n_2}$$

where $n_2 = n_1$. The vector space generated by $\mathfrak{B}_\mathfrak{p} = (X_1, X_2, \dots, X_{n_1})$ is a commutative ideal, the vector space generated by $(A_1, \dots, A_{n_2-1}, A_{n_2})$ is commutative and the matrix representation of $\text{ad}(\sum_{k=1}^{n_2} t_k A_k)$ restricted to \mathfrak{p} with respect to the ordered basis $\mathfrak{B}_\mathfrak{p}$ is

$$N(t) = \left[\text{ad} \left(\sum_{k=1}^{n_2} t_k A_k \right) \Big|_{\mathfrak{p}} \right]_{\mathfrak{B}_\mathfrak{p}} = \begin{bmatrix} t_{n_2} & t_1 & t_2 & \cdots & \cdots & t_{n_2-1} \\ 0 & t_{n_2} & t_1 & t_2 & \cdots & \vdots \\ \vdots & 0 & t_{n_2} & t_1 & \ddots & \\ 0 & \ddots & \ddots & t_{n_2} & \ddots & t_2 \\ 0 & \ddots & 0 & 0 & \ddots & t_1 \\ 0 & \cdots & 0 & 0 & 0 & t_{n_2} \end{bmatrix}. \quad (1.4)$$

Given a linear functional $\lambda = \sum_{k=1}^{n_1} \lambda_k X_k^*$ such that $\lambda_1 \neq 0$, π_λ is an irreducible representation of G modeled as follows. For a fixed Haar measure $d\mu$ on M , π_λ acts on $L^2(M, d\mu)$ as follows

$$\left[\pi_\lambda \left(\exp \left(\sum_{k=1}^{n_2} x_k X_k \right) \exp \left(\sum_{k=1}^{n_2} t_k A_k \right) \right) \mathbf{f} \right] (\mathbf{e}(a)) = e^{2\pi i \langle \exp(N(-a)^*) \lambda, x \rangle} \mathbf{f}(\mathbf{e}(t)^{-1} \mathbf{e}(a)). \quad (1.5)$$

- **(Heisenberg groups and generalizations)** Let $G = P \rtimes M$ be a **step-two** (that is, $[\mathfrak{g}, \mathfrak{g}]$ is non-trivial and is contained in the center of \mathfrak{g}) nilpotent Lie group with Lie algebra spanned by

$$\left\{ \underbrace{X_1, \dots, X_r}_{\mathfrak{z}(\mathfrak{g})}, X_{r+1}, \dots, X_{r+n_2}, A_1, \dots, A_{n_2} \right\}$$

such that

$$P = \exp \left(\sum_{k=1}^{r+n_2} \mathbb{R} X_k \right), M = \exp \left(\sum_{k=1}^{n_2} \mathbb{R} A_k \right)$$

are commutative, and $\exp \mathfrak{z}(\mathfrak{g})$ is the center of G . Moreover, let $\lambda \in \mathfrak{g}^*$ such that

$$\det B(\lambda) = \det \begin{bmatrix} \langle \lambda, [A_1, X_{r+1}] \rangle & \cdots & \langle \lambda, [A_1, X_{r+n_2}] \rangle \\ \vdots & \ddots & \vdots \\ \langle \lambda, [A_{n_2}, X_{r+n_2}] \rangle & \cdots & \langle \lambda, [A_{n_2}, X_{r+n_2}] \rangle \end{bmatrix} \neq 0.$$

Then π_λ is irreducible and is realized as acting on the Hilbert space $L^2(\mathbb{R}^{n_2})$ as follows

$$\left[\pi_\lambda \left(\exp \left(\sum_{k=1}^{r+n_2} x_k X_k \right) \exp \left(\sum_{k=1}^{n_2} t_k A_k \right) \right) \mathbf{f} \right] (a) = e^{2\pi i \left\langle e^{-\left(\sum_{k=1}^{n_2} a_k \text{ad}(A_k) \right) \big|_{\mathfrak{p}}}^* \lambda, x \right\rangle} \mathbf{f}(a - t).$$

- **(Higher order time-frequency groups [30])** Let $G = P \rtimes M$ be a nilpotent Lie group with Lie algebra spanned by $X_1, \dots, X_{n_2+1}, A_1, \dots, A_{n_2}$ such that P and M are commutative closed subgroups and

$$N(t) = \left[\sum_{k=1}^{n_2} t_k \text{ad}(A_k) \Big|_{\mathfrak{p}} \right]_{(X_1, \dots, X_{n_2+1})} = \begin{bmatrix} 0 & t_1 & t_2 & \cdots & t_{n_2} \\ & 0 & t_1 & \ddots & \vdots \\ & & \ddots & \ddots & t_2 \\ & & & 0 & t_1 \\ & & & & 0 \end{bmatrix}.$$

Then $\mathbb{R}X_1 = \mathfrak{z}(\mathfrak{g})$ is the central ideal of the Lie algebra \mathfrak{g} . Let $\lambda = \lambda_1 X_1^* \in \mathfrak{p}^*$ be a linear functional satisfying $\lambda_1 \neq 0$. The corresponding irreducible unitary representation π_λ is realized as acting on the Hilbert space $L^2(\mathbb{R}^{n_2})$ as follows. Given $a = (a_1, \dots, a_{n_1}) \in \mathbb{R}^{n_1}$ and $t = (t_1, \dots, t_{n_1}) \in \mathbb{R}^{n_2}$, we have

$$\left[\pi_\lambda \left(\exp \left(\sum_{k=1}^{n_2+1} x_k X_k \right) \exp \left(\sum_{k=1}^{n_2} t_k A_k \right) \right) \mathbf{f} \right] (a) = e^{2\pi i \langle N(-a)^T \lambda, x \rangle} \mathbf{f}(a - t).$$

- **(Solvable extensions of \mathbb{R}^{n_1})** Let A be a matrix of order n_1 in its Jordan canonical form, and let $G = \mathbb{R}^{n_1} \rtimes \exp(\mathbb{R}A)$ with multiplication law

$$(v, t)(w, s) = (v + \exp(tA)w, t + s).$$

Next, let π_λ be an irreducible representation of G acting on $L^2(\mathbb{R})$ such that

$$[\pi_\lambda(v, s) \mathbf{f}](t) = e^{2\pi i \langle \exp(-tA)^T \lambda, v \rangle} \mathbf{f}(t - s)$$

where $\lambda \in (\mathbb{R}^{n_1})^*$ such that

$$\left\{ \exp(tA)^T \lambda = \lambda : t \in \mathbb{R} \right\}$$

is trivial. Notice that π_λ is neither generally a square-integrable representation nor an integrable representation of G . However, this class of representations fits within the scope of our method. This example is naturally generalized as follows. Let $\mathfrak{m} = \sum_{k=1}^{n_2} \mathbb{R}A_k$ be a solvable Lie algebra of upper-triangular matrices of order n_1 consisting of matrices whose spectrum is contained in \mathbb{R} (see [15, 16, 17].) Next, let $G = \mathbb{R}^{n_1} \rtimes \exp(\mathfrak{m})$ be the semidirect product group with multiplication law given by

$$(v, \exp(A))(v', \exp(A')) = (v + \exp(A)v', \exp(A)\exp(A')).$$

G is a completely solvable Lie group with Lie algebra $\mathbb{R}^{n_1} \oplus \mathfrak{m}$. Moreover, let χ_λ be a character of \mathbb{R}^{n_1} defined by

$$\chi_\lambda(v) = e^{2\pi i \langle \lambda, v \rangle}$$

where $\lambda \in (\mathbb{R}^{n_1})^*$. Furthermore, we assume that

$$G_\lambda = \left\{ \exp A \in \exp \mathfrak{m} : \exp(A)^T \lambda = \lambda \right\}$$

is the trivial subgroup of $\exp \mathfrak{m}$. Next, the representation $\pi_\lambda = \text{ind}_{\mathbb{R}^{n_1}}^G(\chi_\lambda)$ is an irreducible representation of G acting on the Hilbert space $L^2(\exp(\mathfrak{m}), d\mu_{\exp(\mathfrak{m})})$ as follows

$$[\pi_\lambda(\mathbf{x}) \mathbf{f}](\exp A) = \begin{cases} \mathbf{f}(\exp(-A') \exp(A)) & \text{if } \mathbf{x} = (0, \exp A') \\ e^{2\pi i \langle \exp(-A)^T \lambda, v \rangle} \mathbf{f}(\exp(A)) & \text{if } \mathbf{x} = (v, id) \end{cases} \quad (1.6)$$

- **(Extensions of non-commutative nilpotent Lie groups)** Let $G = P \rtimes M$ be a completely solvable Lie group where P is a non-commutative nilpotent Lie normal subgroup and M is isomorphic to a subgroup of the automorphism group of P [4, 35]. Moreover, let us suppose that there exists a character χ_λ of P such that the stabilizer of the coadjoint action of M on the linear functional λ is trivial. Then $\pi_\lambda = \text{ind}_P^{P \rtimes M}(\chi_\lambda)$ is irreducible and given $\mathbf{f} \in L^2(M, d\mu_M)$,

$$[\pi_\lambda(p, n) \mathbf{f}](m) = e^{2\pi i \langle \text{Ad}(m^{-1})^* \lambda, \log(p) \rangle} \mathbf{f}(n^{-1}m)$$

as described in (1.3)

1.4 Short overview of the main results

The main objective of the present paper is to establish the following. Let G and π_λ be as defined in Condition 1. There exist a discrete subset Γ of G and a function $\mathbf{f} \in L^2(M, d\mu_M)$ such that $\{\pi_\lambda(\gamma) \mathbf{f} : \gamma \in \Gamma\}$ is a frame for $L^2(M, d\mu_M)$. Moreover, the function \mathbf{f} can be chosen to be infinitely smooth and compactly supported on M . In contrast to other discretization schemes such as the coorbit theory, we insist that **no assumption** is being made about the integrability of π_λ .

1.4.1 A unified procedure for the construction of tight frames

In this subsection, we shall present a scheme which is systematically exploited to construct pairs (Γ, \mathbf{f}) such that $\{\pi_\lambda(\gamma) \mathbf{f} : \gamma \in \Gamma\}$ is a tight frame for $L^2(M, d\mu_M)$. Our procedure is outlined as follows.

1. Fix a linear functional λ with corresponding unitary irreducible representation

$$\pi_\lambda = \text{ind}_P^{P \rtimes M}(\chi_\lambda).$$

Given $a = (a_1, a_2, \dots, a_{n_2}) \in \mathbb{R}^m$, let

$$A(a_1, a_2, \dots, a_{n_2}) = A(a) = \sum_{k=1}^{n_2} a_k A_k = A.$$

Next, let $\theta_\lambda : \mathfrak{m} \rightarrow \mathfrak{p}^*$ be a smooth function defined as follows

$$\theta_\lambda \left(\sum_{k=1}^{n_2} a_k A_k \right) = \mathbf{C}(a)^* \lambda = \left(\left(e^{-\text{ad}(a_{n_2} A_{n_2})} \dots e^{-\text{ad}(a_2 A_2)} e^{-\text{ad}(a_1 A_1)} \right) \Big|_{\mathfrak{p}} \right)^* \lambda. \quad (1.7)$$

Then θ_λ represents the coadjoint action of the conormal subgroup M on the linear functional λ . Moreover, under the assumptions stated above, θ_λ defines an immersion of \mathfrak{m} into \mathfrak{p}^* (see Lemma 10). In other words, the differential of θ_λ is injective at each point A , and θ_λ behaves locally like an injective function. Let D_{θ_λ} be the differential of θ_λ at the zero element in \mathfrak{m} . To be more explicitly, if

$$\text{Jac}_{\theta_\lambda}(a) = \begin{bmatrix} \frac{\partial [(\mathbf{C}(a)^* \lambda)_1]}{\partial a_1} & \dots & \frac{\partial [(\mathbf{C}(a)^* \lambda)_1]}{\partial a_{n_2}} \\ \vdots & \ddots & \vdots \\ \frac{\partial [(\mathbf{C}(a)^* \lambda)_{n_1}]}{\partial a_1} & \dots & \frac{\partial [(\mathbf{C}(a)^* \lambda)_{n_1}]}{\partial a_{n_2}} \end{bmatrix} \quad (1.8)$$

then

$$D_{\theta_\lambda} = \text{Jac}_{\theta_\lambda}(0). \quad (1.9)$$

2. Next, let $D_{\theta_\lambda}(j_1, \dots, j_{n_2})$ be the submatrix of D_{θ_λ} obtained by retaining the j_1^{th} -row, \dots , $j_{n_2}^{\text{th}}$ -row of the matrix $[D_{\theta_\lambda}]_{\mathfrak{B}_p}$. Put

$$\mathcal{T} = \{\mathbf{I} : \mathbf{I} = (j_1, \dots, j_{n_2}) \text{ and } 1 \leq j_1 < \dots < j_{n_2} \leq n_1\}$$

and define

$$\mathcal{A} = \{D_{\theta_\lambda}(\mathbf{I}) : \mathbf{I} \in \mathcal{T} \text{ and } \det(D_{\theta_\lambda}(\mathbf{I})) \neq 0\}.$$

Note that if $[D_{\theta_\lambda}]_{\mathfrak{B}_p}$ is an invertible matrix of order n_2 (this is not generally the case) then \mathcal{A} is necessarily a singleton. Fix

$$\mathbf{J}(\lambda) = (j_1, \dots, j_{n_2}) \in \mathcal{T} \quad (1.10)$$

such that $D_{\theta_\lambda}(\mathbf{J}(\lambda)) \in \mathcal{A}$ and define $\mathbf{P}_{\mathbf{J}(\lambda)} : \mathfrak{p} \rightarrow \mathfrak{p}$ such that

$$\mathbf{P}_{\mathbf{J}(\lambda)}(X_k) = \begin{cases} X_k & \text{if } k \in \mathbf{J}(\lambda) \\ 0 & \text{if } k \notin \mathbf{J}(\lambda) \end{cases}. \quad (1.11)$$

Clearly, $\mathbf{P}_{\mathbf{J}(\lambda)}$ is a linear map, and the matrix representation of the linear map $\mathbf{P}_{\mathbf{J}(\lambda)}$ is a diagonal matrix whose spectrum is contained in the discrete set $\{0, 1\}$. Evidently, $\mathbf{P}_{\mathbf{J}(\lambda)}$ defines an orthogonal projection of rank n_2 on the Lie algebra \mathfrak{p} . Furthermore, if X belongs to the range of $\mathbf{P}_{\mathbf{J}(\lambda)}$, we obtain

$$[\pi_\lambda(\exp X) \mathbf{f}](\exp(a_1 A_1) \cdots \exp(a_{n_2} A_{n_2})) = e^{2\pi i \mathbf{Q}(a, \lambda, X)} \mathbf{f}(\exp(a_1 A_1) \cdots \exp(a_{n_2} A_{n_2})) \quad (1.12)$$

where

$$\mathbf{Q}(a, \lambda, X) = \langle \mathbf{P}_{\mathbf{J}(\lambda)}^*(\mathbf{C}(a_1, a_2, \dots, a_{n_2}))^* \lambda, X \rangle.$$

3. Define $\beta_{\mathbf{J}(\lambda)} : \mathfrak{m} \rightarrow \mathbf{P}_{\mathbf{J}(\lambda)}^*(\mathfrak{p}^*)$ such that

$$\beta_{\mathbf{J}(\lambda)}\left(\sum_{k=1}^{n_2} a_k A_k\right) = \mathbf{P}_{\mathbf{J}(\lambda)}^*(\mathbf{C}(a_1, a_2, \dots, a_{n_2}))^* \lambda. \quad (1.13)$$

According to Lemma 12, $\beta_{\mathbf{J}(\lambda)}$ is a local diffeomorphism at the zero element in \mathfrak{m} . By the Inverse Function Theorem ([23], Theorem 5.11) there exists a connected open subset \mathcal{O} around the zero element of \mathfrak{m} such that the restriction of $\beta_{\mathbf{J}(\lambda)}$ to \mathcal{O} is a diffeomorphism. We shall coin the collection of maps

$$\mathfrak{Data}_{(\pi_\lambda, G)} = \{\beta_{\mathbf{J}(\lambda)} : D_{\theta_\lambda}(\mathbf{J}(\lambda)) \in \mathcal{A}\}, \quad (1.14)$$

the **orbital data corresponding to the linear functional** λ . Next, let

$$\mathbf{L} = \left\{ s \in (0, \infty) : \sum_{k=1}^{n_2} \left[-\frac{s}{2}, \frac{s}{2}\right] A_k \subset \mathcal{O} \right\}. \quad (1.15)$$

Clearly, \mathbf{L} is a non-empty subset of $(0, \infty)$. Fix $\epsilon \in \mathbf{L}$, define a relatively compact subset $\Omega_\epsilon \subset M$ and a discrete subset Γ_M^ϵ of M as follows.

$$\Omega_\epsilon = \exp\left(\left[-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right] A_1\right) \cdots \exp\left(\left[-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right] A_{n_2}\right), \quad (1.16)$$

and

$$\Gamma_M^\epsilon = \exp(\epsilon \mathbb{Z} A_1) \cdots \exp(\epsilon \mathbb{Z} A_{n_2})$$

respectively. Appealing to Proposition 13, the collection

$$\{\gamma^{-1} \Omega_\epsilon : \gamma \in \Gamma_M^\epsilon\}$$

is a measurable partition of M . Next, define

$$\mathcal{O}_\epsilon = \sum_{k=1}^{n_2} \left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) A_k \text{ and } F_\epsilon = \sum_{k=1}^{n_2} \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right] A_k. \quad (1.17)$$

Note that \mathcal{O}_ϵ is just the interior of the set F_ϵ , and the map $\beta_{\mathbf{J}(\lambda)}$ is continuous on \mathcal{O} and uniformly continuous on $\overline{\mathcal{O}_\epsilon}$ (the topological closure of \mathcal{O}_ϵ .) Moreover, $\beta_{\mathbf{J}(\lambda)}(\overline{\mathcal{O}_\epsilon})$ is a compact subset of

$$\mathbf{P}_{\mathbf{J}(\lambda)}^*(\mathfrak{p}^*) = \sum_{j \in \mathbf{J}(\lambda)} \mathbb{R}X_j^*.$$

Thus, $\beta_{\mathbf{J}(\lambda)}(\overline{\mathcal{O}_\epsilon})$ is a bounded subset, has positive Lebesgue measure on $\mathbf{P}_{\mathbf{J}(\lambda)}^*(\mathfrak{p}^*)$, and the set $\beta_{\mathbf{J}(\lambda)}(F_\epsilon)$ is contained in a fundamental domain of a full-rank lattice of the vector space $\mathbf{P}_{\mathbf{J}(\lambda)}^*(\mathfrak{p}^*)$.

4. Let

$$\mathbf{T}^\epsilon = \left\{ \mathcal{T} : \mathcal{T} \text{ is a full-rank lattice of } \mathbf{P}_{\mathbf{J}(\lambda)}^*(\mathfrak{p}^*) \text{ and } \sum_{\kappa \in \mathcal{T}} \mathbf{1}_{\beta_{\mathbf{J}(\lambda)}(F_\epsilon)}(\xi + \kappa) \leq 1 \text{ for every } \xi \in \mathbf{P}_{\mathbf{J}(\lambda)}^*(\mathfrak{p}^*) \right\}. \quad (1.18)$$

In other words, $\mathcal{L} \in \mathbf{T}^\epsilon$ if and only if $\beta_{\mathbf{J}(\lambda)}(F_\epsilon)$ is contained in a fundamental domain of $\beta_{\mathbf{J}(\lambda)}(F_\epsilon)$. It is shown in Lemma 14 that \mathbf{T}^ϵ is a non-empty set. To be more specific, Lemma 14 describes explicitly an invertible linear map

$$\mathcal{L}^\epsilon : \sum_{j \in \mathbf{J}(\lambda)} \mathbb{R}X_j \rightarrow \sum_{j \in \mathbf{J}(\lambda)} \mathbb{R}X_j \quad (1.19)$$

such that given

$$\Gamma_P^\epsilon = \exp(\Lambda_\epsilon) \subset P \text{ where } \Lambda_\epsilon = \mathbb{Z}\text{-span}\{\mathcal{L}^\epsilon X_j : j \in \mathbf{J}(\lambda)\}$$

the following holds true. The set

$$\beta_{\mathbf{J}(\lambda)}(F_\epsilon) \subset \sum_{j \in \mathbf{J}(\lambda)} \mathbb{R}X_j^*$$

is contained in a Lebesgue measurable fundamental domain of

$$\Lambda_\epsilon^* = \mathbb{Z}\text{-span}\left\{(\mathcal{L}^\epsilon)^\top X_j^* : j \in \mathbf{J}(\lambda)\right\} \subset \mathfrak{p}^*$$

where $(\mathcal{L}^\epsilon)^\top$ is the transpose inverse of \mathcal{L}^ϵ .

5. For a positive measurable function r defined on M , we define $\mathbf{f}_{r,\epsilon} = r \times \mathbf{1}_{\Omega_\epsilon}$. Next, let

$$\mathbf{e} \left(\sum_{k=1}^{n_2} a_k A_k \right) = \exp(a_1 A_1) \cdots \exp(a_{n_2} A_{n_2}).$$

Clearly $\mathbf{e} : \mathfrak{m} \rightarrow M$ is a diffeomorphism. Furthermore, let ρ be the Radon-Nikodym derivative given by

$$d\mu_M \left(\mathbf{e} \left(\sum_{k=1}^{n_2} a_k A_k \right) \right) = d\mu_M (\exp(a_1 A_1) \cdots \exp(a_{n_2} A_{n_2})) = \rho(A) dA \quad (1.20)$$

where $d\mu_M$ is a left Haar measure on the solvable group M and dA is the Lebesgue measure on $\mathfrak{m} = \mathbb{R}^{n_2}$. The function ρ is an analytic function which is explicitly computed as follows. Let ν be a smooth bijection defined on the Lie algebra \mathfrak{m} such that

$$\exp(a_1 A_1) \cdots \exp(a_{n_2} A_{n_2}) = \exp\left(\nu\left(\sum_{k=1}^{n_2} a_k A_k\right)\right) = \exp\left(\sum_{k=1}^{n_2} \nu_k(a) A_k\right). \quad (1.21)$$

For example, if \mathfrak{m} is commutative then ν is the identity map, and if \mathfrak{m} is a nilpotent algebra, then ν is a polynomial. In general, $\exp(a_1 A_1) \cdots \exp(a_{n_2} A_{n_2})$ is computed by applying the Campbell-Baker-Hausdorff formula iteratively as follows. Defining

$$\begin{aligned} X * Y &= \sum_{n \geq 0} \frac{(-1)^{n+1}}{n} \sum_{p_i + q_i > 0, 1 \leq i \leq n} \frac{(\sum_{i=1}^n (p_i + q_i))^{-1}}{p_1! q_1! \cdots p_n! q_n!} \\ &\quad (adX)^{p_1} (adY)^{q_1} \cdots (adX)^{p_n} (adY)^{q_n-1} Y \\ &= \nu(X + Y), \end{aligned}$$

then

$$\exp(a_1 A_1) \cdots \exp(a_{n_2} A_{n_2}) = \exp(a_1 A_1 * \cdots * a_{n_2} A_{n_2}).$$

In other words,

$$\nu\left(\sum_{k=1}^{n_2} a_k A_k\right) = a_1 A_1 * \cdots * a_{n_2} A_{n_2}$$

and for a positive function $\mathbf{f} \in L^1(M, d\mu_M)$,

$$\begin{aligned} \int_M \mathbf{f}(m) d\mu_M(m) &= \int_{\mathfrak{m}} \mathbf{f}(\exp A) d\mu_M(\exp A) \\ &= \int_{\mathfrak{m}} \mathbf{f}(\exp A) \underbrace{\left| \det \left(\frac{id - e^{-\text{ad}(A)}}{\text{ad}(A)} \right) \right|}_{=w(A)} dA \\ &= \int_{\mathfrak{m}} \mathbf{f}(\exp A) w(A) dA \\ &= \int_{\mathfrak{m}} \mathbf{f}(\exp \nu(A)) \overbrace{w(\nu(A)) d(\nu(A))}^{=\rho(A)dA} \\ &= \int_{\mathfrak{m}} \mathbf{f}(\exp \nu(A)) \rho(A) dA. \end{aligned}$$

6. Fix $\Lambda_\epsilon^* \in \mathbf{T}^\epsilon$. Let

$$\begin{aligned} \Gamma^\epsilon &= (\Gamma_M^\epsilon)^{-1} \Gamma_P^\epsilon = (\exp(\epsilon \mathbb{Z} A_1) \cdots \exp(\epsilon \mathbb{Z} A_{n_2}))^{-1} \exp(\Lambda_\epsilon) \\ &= \exp(\epsilon \mathbb{Z} A_{n_2}) \cdots \exp(\epsilon \mathbb{Z} A_1) \exp\left(\sum_{j \in J(\lambda)} \mathcal{L}^\epsilon \mathbb{Z} X_j\right). \end{aligned}$$

Define the system

$$\mathcal{S}(\mathbf{f}_{r,\epsilon}, \Gamma^\epsilon) = \{\pi_\lambda(\kappa) \mathbf{f}_{r,\epsilon} : \kappa \in \Gamma^\epsilon\} \quad (1.22)$$

and let $\Theta_\lambda(\xi)$ be the absolute value of the determinant of the Jacobian of $[\beta_{\mathbf{J}(\lambda)}|_{\mathcal{O}}]^{-1}$.

Theorem 2 *If $r(\mathbf{e}(A)) = (\rho(A) \times \Theta_\lambda(\beta_{\mathbf{J}(\lambda)}(A)))^{-1/2}$ then $\mathbf{f}_{r,\epsilon}$ is square-integrable with respect to the Haar measure $d\mu_M$. Moreover, the system $\mathcal{S}(\mathbf{f}_{r,\epsilon}, \Gamma^\epsilon)$ is a tight frame for $L^2(M, d\mu_M)$ with frame bound $|\det(\mathcal{L}^\epsilon)|^{-1}$. Consequently, $\mathcal{S}(|\det(\mathcal{L}^\epsilon)|^{1/2} \mathbf{f}_{r,\epsilon}, \Gamma^\epsilon)$ is a Parseval frame for $L^2(M, d\mu_M)$.*

1.4.2 Constructions of smooth frames of compact supports

We shall now present an explicit construction of a frame $\mathcal{S}(\mathbf{s}, \Gamma)$ such that \mathbf{s} is smooth and compactly supported. To this end, we proceed as follows.

1. We fix $\epsilon \in \mathbf{L}$. Next, define Ω_ϵ° to be an open subset of M such that

$$\Omega_\epsilon^\circ = \exp\left(\left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) A_1\right) \cdots \exp\left(\left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) A_{n_2}\right). \quad (1.23)$$

As observed above, the restriction of $\beta_{\mathbf{J}(\lambda)}$ to the open set

$$\mathcal{O}_\epsilon = \left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) A_1 + \cdots + \left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) A_{n_2} \subset \mathfrak{m} \quad (1.24)$$

defines a diffeomorphism between \mathcal{O}_ϵ and $\beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon)$.

2. We define the map

$$\Phi_{\mathbf{J}(\lambda)}^\epsilon : \Omega_\epsilon^\circ \rightarrow \beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon)$$

such that

$$\Phi_{\mathbf{J}(\lambda)}^\epsilon(\mathbf{e}(a_1 A_1 + \cdots + a_{n_2} A_{n_2})) = \beta_{\mathbf{J}(\lambda)}(a_1 A_1 + \cdots + a_{n_2} A_{n_2}). \quad (1.25)$$

3. Let $\mathbf{s} \in C_c^\infty(M)$ such that the support of \mathbf{s} is a compact subset of Ω_ϵ° . Then the support of the function $\mathbf{s} \circ [\Phi_{\mathbf{J}(\lambda)}^\epsilon]^{-1}$ is a compact subset $\Sigma_{\mathbf{s}}$ of $\beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon)$. Put

$$\Lambda_\epsilon = \sum_{j \in \mathbf{J}(\lambda)} \mathbb{Z} \mathcal{L}^\epsilon X_j$$

and let

$$\Gamma_P^\epsilon = \exp(\Lambda_\epsilon) = \exp\left(\sum_{j \in \mathbf{J}(\lambda)} \mathbb{Z} \mathcal{L}^\epsilon X_j\right) \quad (1.26)$$

be a discrete subset of P such that $\Sigma_{\mathbf{s}}$ is contained in a fundamental domain of a lattice

$$\Lambda_\epsilon^* = \sum_{j \in \mathbf{J}(\lambda)} \mathbb{Z} (\mathcal{L}^\epsilon)^\top X_j^*. \quad (1.27)$$

4. Let $d\xi$ be the canonical Lebesgue measure defined on $\beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon)$. Define the Radon-Nikodym derivative $\Psi_{\mathbf{J}(\lambda)}^\epsilon$ such that

$$\Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi) d\xi = d\mu_M \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right). \quad (1.28)$$

Then $\Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi)$ is a positive smooth function defined on $\beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon)$. Moreover, it is proved in Proposition 24 that the function

$$m \mapsto \left(\Psi_{\mathbf{J}(\lambda)}^\epsilon \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(m) \right) \right)^{-1}$$

is integrable with respect to the Haar measure on M on any compact subset of Ω_ϵ° . Furthermore, the auxiliary function Υ given by

$$\Upsilon(m) = \sqrt{\Psi_{\mathbf{J}(\lambda)}^\epsilon \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(m) \right) \left| \det(\mathcal{L}^\epsilon)^\top \right|} \mathbf{s}(m) \quad (1.29)$$

is a smooth function which is compactly supported on M . Consequently, there exists a sufficiently dense discrete subset

$$\Gamma_M^\epsilon(\mathbf{s}) = \Gamma_M^\epsilon \subset M \quad (1.30)$$

depending on the function \mathbf{s} such that

$$\inf \left\{ \sum_{\gamma \in \Gamma_M^\epsilon} |\Upsilon(\gamma m)|^2 : m \in M \right\} = A_{\mathbf{s}, \Gamma_M^\epsilon, \Lambda^\star} > 0 \quad (1.31)$$

and

$$\sup \left\{ \sum_{\gamma \in \Gamma_M^\epsilon} |\Upsilon(\gamma m)|^2 : m \in M \right\} = B_{\mathbf{s}, \Gamma_M^\epsilon, \Lambda^\star} < \infty. \quad (1.32)$$

Finally, let

$$\Gamma^\epsilon = (\Gamma_M^\epsilon)^{-1} \Gamma_P^\epsilon.$$

Theorem 3 *Let \mathbf{s} and Γ^ϵ be as defined above. Then the system $\mathcal{S}(\mathbf{s}, \Gamma^\epsilon)$ is a frame for $L^2(M, \mu_M)$ with optimal lower and upper frame bounds $A_{\mathbf{s}, \Gamma_M^\epsilon, \Lambda^\star}$ and $B_{\mathbf{s}, \Gamma_M^\epsilon, \Lambda^\star}$ respectively.*

2 A toy example

In order to set the stage for the generalization to come, we shall present a toy example which illustrates the core ideas of our scheme. Let $G = P \rtimes M$ be a simply connected, connected completely solvable Lie group with Lie algebra spanned by (X_1, X_2, A_1) such that $P = \exp(\mathbb{R}X_1 + \mathbb{R}X_2)$ and $M = \exp \mathbb{R}A_1$ are commutative closed subgroups of G and

$$[A_1, X_2] = X_1, [A_1, X_1] = X_2 + X_1.$$

Thus,

$$[ad(aA_1)]_{(X_1, X_2)} = \begin{bmatrix} a & a \\ 0 & a \end{bmatrix} \text{ and } e^{[ad(aA_1)]_{(X_1, X_2)}} = \begin{bmatrix} e^a & ae^a \\ 0 & e^a \end{bmatrix}. \quad (2.1)$$

Fixing $\lambda = X_1^*$, the orbital data corresponding to λ is equal to

$$\mathbf{Data}_{(\pi_\lambda, G)} = \{a \mapsto \beta_{(1)}(a) = e^{-a}, a \mapsto \beta_{(2)}(a) = -ae^{-a}\}.$$

In order to simplify our presentation, we identify G with the semi-direct product group $\mathbb{R}^2 \rtimes \mathbb{R}$ with multiplication law given by

$$(x_1, x_2, a)(y_1, y_2, b) = (x_1 + e^a(y_1 + ay_2), x_2 + e^a y_2, a + b).$$

Indeed, the identification above is valid since the mapping

$$(x_1, x_2, a) \mapsto \exp(x_1 X_1 + x_2 X_2) \exp(a A_1)$$

is a Lie group isomorphism. We realize the representation π_λ as acting in $L^2(\mathbb{R})$ as follows

$$[\pi_\lambda(x_1, x_2, a)\mathbf{f}](t) = \exp\left(2\pi i \left\langle \begin{bmatrix} e^{-t} \\ -te^{-t} \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle\right) \mathbf{f}(t - a). \quad (2.2)$$

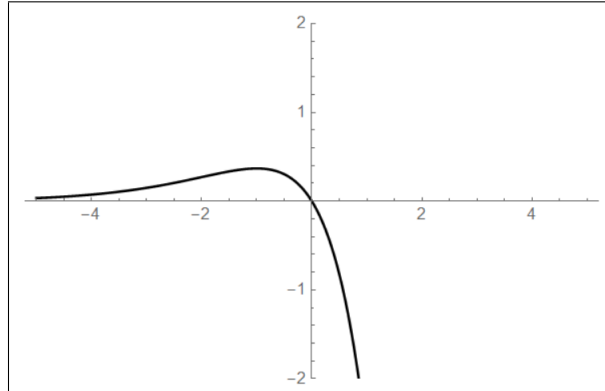
Referring back to the orbital data described previously, we fix $\mathbf{J}(\lambda) = (2)$ and we define

$$\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right).$$

Then

$$[\pi_\lambda(0, x_2, a)\mathbf{f}](t) = \exp(2\pi i \beta_{(2)}(t)x_2) \mathbf{f}(t - a) \text{ and } \beta_{(2)}(t) = -te^{-t}.$$

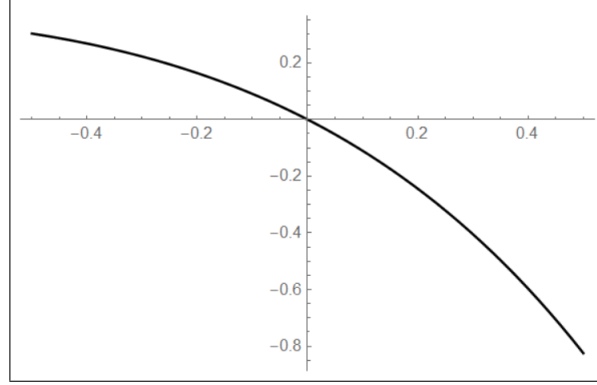
The reader might find the following remarks instructive. First, the function $\beta_{(2)}(t) = -te^{-t}$ does not define a global diffeomorphism between \mathbb{R} and its range. Indeed, from the graph below, it is clear that the map $t \mapsto \beta_{(2)}(t)$ is not injective



However, since the derivative of $\beta_{(2)}(t)$ does not vanish at zero, there exists an open set \mathcal{O} around zero such that the restriction of $\beta_{(2)}$ to \mathcal{O} defines a diffeomorphism between \mathcal{O} and $\beta_{(2)}(\mathcal{O})$. Indeed,

$$\beta_{(2)}|_{(-\frac{1}{2}, \frac{1}{2})} : \left(-\frac{1}{2}, \frac{1}{2}\right) \rightarrow \beta_{(2)}\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right)$$

is a diffeomorphism



Let $\mathbf{A} = (-\frac{1}{2}, \frac{1}{2})$ be the interior of Ω . Secondly, let Θ_λ be defined as follows

$$\Theta_\lambda(\xi) = \left| \left(\frac{d[\beta_{(2)}|_{\mathbf{A}}]^{-1}(\xi)}{d\xi} \right) (\xi) \right| = \left| \frac{[\beta_{(2)}|_{\mathbf{A}}]^{-1}(\xi)}{\xi \left(1 + [\beta_{(2)}|_{\mathbf{A}}]^{-1}(\xi) \right)} \right|. \quad (2.3)$$

It can be shown numerically that

$$t \mapsto \sqrt{\frac{1}{\Theta_\lambda(-te^{-t})}} \in L^2(\mathbf{A}, dt).$$

Note that

$$\mathbf{B} = \beta_{(2)}(\mathbf{A}) = \left(-\frac{1}{2\sqrt{e}}, \frac{\sqrt{e}}{2} \right)$$

and the Lebesgue measure of \mathbf{B} which we denote by $|\mathbf{B}|$ is equal to

$$\frac{\sqrt{e}}{2} + \frac{1}{2\sqrt{e}} = \frac{1+e}{2\sqrt{e}}.$$

Consequently, \mathbf{B} is up to a null set a fundamental domain of the lattice $\frac{1+e}{2\sqrt{e}}\mathbb{Z}$. Thus, the trigonometric system

$$\left\{ \xi \mapsto \frac{\exp(2\pi i \xi k)}{\left(\frac{1+e}{2\sqrt{e}} \right)^{1/2}} : k \in \frac{2\sqrt{e}}{1+e}\mathbb{Z} \right\}$$

is an orthonormal basis for the Hilbert space $L^2(\mathbf{B}, d\xi)$. Put $c = \frac{2\sqrt{e}}{1+e}$ and

$$\mathbf{f}(t) = \frac{\mathbf{1}_\Omega(t)}{(\Theta_\lambda(\beta_{(2)}(t)))^{1/2}} = \frac{\mathbf{1}_\Omega(t)}{\sqrt{\Theta_\lambda(-te^{-t})}}.$$

Next, we shall prove that the system

$$\left\{ \frac{\mathbf{1}_\Omega(t) e^{2\pi i (e^{-t}t)k}}{\sqrt{\Theta_\lambda(-te^{-t})}} : k \in c\mathbb{Z} \right\} = \{ \pi_\lambda(\gamma) \mathbf{f} : \gamma \in \Gamma_M \}$$

is a tight frame for $L^2(\Omega)$ with frame bounds c^{-1} . Indeed, given $\mathbf{h} \in L^2(\Omega)$ we have

$$\sum_{k \in c\mathbb{Z}} \left| \langle \mathbf{h}, \pi_\lambda(0, k, 0) \mathbf{f} \rangle_{L^2(\Omega)} \right|^2 = \sum_{k \in c\mathbb{Z}} \left| \int_{\mathbf{A}} \frac{\mathbf{h}(t) e^{2\pi i(e^{-t}t)k}}{(\Theta_\lambda(\beta_{(2)}(t)))^{1/2}} dt \right|^2.$$

The change of variable $t = (\beta_{J(\lambda)})^{-1}(\xi)$ allows us to proceed as follows

$$\begin{aligned} & \sum_{k \in c\mathbb{Z}} \left| \langle \mathbf{h}, \pi_\lambda(0, k, 0) \mathbf{f} \rangle_{L^2(\mathbf{A})} \right|^2 \\ &= \sum_{k \in c\mathbb{Z}} \left| \int_{\mathbf{B}} \left[\frac{\mathbf{h}(\beta_{J(\lambda)}^{-1}(\xi)) e^{-2\pi i \xi k}}{(\Theta_\lambda(\beta_{(2)}(\beta_{(2)}^{-1}(\xi))))^{1/2}} \right] d((\beta_{(2)})^{-1}(\xi)) \right|^2 \\ &= \sum_{k \in c\mathbb{Z}} \left| \int_{\mathbf{B}} \left[\frac{\mathbf{h}(\beta_{(2)}^{-1}(\xi)) e^{-2\pi i \xi k} \Theta_\lambda(\xi)}{(\Theta_\lambda(\xi))^{1/2}} \right] d\xi \right|^2 \\ &= \sum_{k \in c\mathbb{Z}} \left| \int_{\mathbf{B}} \left[\frac{\mathbf{h}(\beta_{(2)}^{-1}(\xi)) \Theta_\lambda(\xi)}{(\Theta_\lambda(\xi))^{1/2}} \right] e^{-2\pi i \xi k} d\xi \right|^2 \\ &= \sum_{k \in c\mathbb{Z}} \left| \int_{\mathbf{B}} \left[\frac{\left(\frac{1+e}{2\sqrt{e}}\right)^{1/2} \mathbf{h}(\beta_{(2)}^{-1}(\xi)) \Theta_\lambda(\xi)}{(\Theta_\lambda(\xi))^{1/2}} \right] \frac{e^{-2\pi i(\xi k)}}{\left(\frac{1+e}{2\sqrt{e}}\right)^{1/2}} d\xi \right|^2. \end{aligned}$$

Put

$$\mathbf{H}(\xi) = \frac{\left(\frac{1+e}{2\sqrt{e}}\right)^{1/2} \mathbf{h}(\beta_{(2)}^{-1}(\xi)) \Theta_\lambda(\xi)}{(\Theta_\lambda(\xi))^{1/2}}. \quad (2.4)$$

Then

$$\sum_{k \in c\mathbb{Z}} \left| \langle \mathbf{h}, \pi_\lambda(0, k, 0) \mathbf{f} \rangle_{L^2(\Omega)} \right|^2 = \sum_{k \in c\mathbb{Z}} \left| \int_{\mathbf{B}} \mathbf{H}(\xi) \left(\frac{e^{-2\pi i \xi k}}{\left(\frac{1+e}{2\sqrt{e}}\right)^{1/2}} \right) d\xi \right|^2.$$

Since

$$\left\{ \xi \mapsto \frac{e^{-2\pi i(\xi k)} \mathbf{1}_{\left(-\frac{1}{2\sqrt{e}}, \frac{\sqrt{e}}{2}\right)}}{\left(\frac{1+e}{2\sqrt{e}}\right)^{1/2}} : k \in c\mathbb{Z} \right\}$$

is an orthonormal basis for $L^2(\mathbf{B})$ it follows that

$$\begin{aligned} & \sum_{k \in c\mathbb{Z}} \left| \langle \mathbf{h}, \pi_\lambda(0, k, 0) \mathbf{f} \rangle_{L^2(\Omega)} \right|^2 = \int_{\mathbf{B}} |\mathbf{H}(\xi)|^2 d\xi \\ &= \int_{\mathbf{B}} \left| \frac{\left(\frac{1+e}{2\sqrt{e}}\right)^{1/2} \mathbf{h}(\beta_{(2)}^{-1}(\xi)) \Theta_\lambda(\xi)}{(\Theta_\lambda(\xi))^{1/2}} \right|^2 d\xi = (*) \end{aligned}$$

and

$$\begin{aligned}
(*) &= c^{-1} \int_{\mathbf{B}} \left| \mathbf{h} \left(\beta_{(2)}^{-1}(\xi) \right) (\Theta_{\lambda}(\xi))^{1/2} \right|^2 d\xi \\
&= c^{-1} \int_{\mathbf{B}} \left| \mathbf{h} \left(\beta_{(2)}^{-1}(\xi) \right) \right|^2 \underbrace{\Theta_{\lambda}(\xi) d\xi}_{=d(\beta_{(2)}^{-1}(\xi))} \\
&= c^{-1} \int_{\mathbf{B}} \left| \mathbf{h} \left(\beta_{(2)}^{-1}(\xi) \right) \right|^2 d(\beta_{(2)}^{-1}(\xi)).
\end{aligned}$$

Next, the change of variable $\xi = \beta_{\mathbf{J}(\lambda)}(t)$ yields

$$\begin{aligned}
&\sum_{k \in c\mathbb{Z}} \left| \langle \mathbf{h}, \pi_{\lambda}(0, k, 0) \mathbf{f} \rangle_{L^2(\mathbf{A})} \right|^2 \\
&= c^{-1} \int_{\mathbf{A}} \left| \mathbf{h} \left(\beta_{(2)}^{-1}(\beta_{(2)}(t)) \right) \right|^2 d(\beta_{(2)}^{-1}(\beta_{(2)}(t))) \\
&= c^{-1} \int_{\mathbf{A}} |\mathbf{h}(t)|^2 dt.
\end{aligned}$$

As such,

$$\left\{ \frac{\mathbf{1}_{\Omega}(t) e^{2\pi i(e^{-t}t)k}}{(\Theta_{\lambda}(\beta_{(2)}(t)))^{1/2}} : k \in c\mathbb{Z} \right\}$$

is a tight frame for $L^2(\mathbf{A})$ with frame bounds $c^{-1} = (1+e)(2\sqrt{e})^{-1}$. Next, we observe that

$$\left\{ \left[\frac{1}{2} - \ell, \frac{1}{2} - \ell \right) : \ell \in \mathbb{Z} \right\}$$

is a measurable partition of the real line. Thus, the collection

$$\left\{ t \mapsto \pi_{\lambda}((0, 0, \ell)(0, k, 0)) \frac{\mathbf{1}_{\Omega}(t)}{(\Theta_{\lambda}(\beta_{(2)}(t)))^{1/2}} : k \in \frac{2\sqrt{e}}{1+e}\mathbb{Z}, \ell \in \mathbb{Z} \right\}$$

is a tight frame with frame bound c^{-1} . In conclusion, the family of vectors

$$\pi_{\lambda}(\exp(\mathbb{Z}A_1) \exp((1+e)^{-1}(2\sqrt{e})\mathbb{Z}X_2)) \frac{\mathbf{1}_{\Omega}(\cdot)}{(\Theta_{\lambda}(\beta_{(2)}(\cdot)))^{1/2}} \quad (2.5)$$

is a tight frame with frame bound $(1+e)(2\sqrt{e})^{-1}$ for $L^2(\mathbb{R})$.

3 Intermediate results

In order to help the reader keep track of the various objects introduced in our scheme, we provide the following tables.

$\theta_\lambda \left(\sum_{k=1}^{n_2} a_k A_k \right) = \mathbf{C}(a)^* \lambda = \left(\left(e^{-\text{ad}(a_{n_2} A_{n_2})} \dots e^{-\text{ad}(a_2 A_2)} e^{-\text{ad}(a_1 A_1)} \right) \Big _{\mathfrak{p}} \right)^* \lambda$
$D_{\theta_\lambda} = \text{Jac}_{\theta_\lambda}(0), \mathcal{T} = \{\mathbf{I} : \mathbf{I} = (j_1, \dots, j_{n_2}) \text{ and } 1 \leq j_1 < \dots < j_{n_2} \leq n_1\}$
$\mathcal{A} = \{D_{\theta_\lambda}(\mathbf{I}) : \mathbf{I} \in \mathcal{T} \text{ and } \det(D_{\theta_\lambda}(\mathbf{I})) \neq 0\}$
Fixing $\mathbf{J}(\lambda) = (j_1, \dots, j_{n_2}) \in \mathcal{T}$, $\mathbf{P}_{\mathbf{J}(\lambda)}(X_k) = \begin{cases} X_k & \text{if } k \in \mathbf{J}(\lambda) \\ 0 & \text{if } k \notin \mathbf{J}(\lambda) \end{cases}$
$\beta_{\mathbf{J}(\lambda)} \left(\sum_{k=1}^{n_2} a_k A_k \right) = \mathbf{P}_{\mathbf{J}(\lambda)}^* (\mathbf{C}(a_1, a_2, \dots, a_{n_2}))^* \lambda$
$\mathfrak{Data}_{(\pi_\lambda, G)} = \{\beta_{\mathbf{J}(\lambda)} : D_{\theta_\lambda}(\mathbf{J}(\lambda)) \in \mathcal{A}\}, \mathbf{L} = \{s \in (0, \infty) : \sum_{k=1}^{n_2} \left[-\frac{s}{2}, \frac{s}{2} \right] A_k \subset \mathcal{O}\}$
ϵ is a fixed positive number in \mathbf{L} , $\Omega_\epsilon = \exp \left(\left[-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right] A_1 \right) \dots \exp \left(\left[-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right] A_{n_2} \right)$
$\Gamma_M^\epsilon = \exp(\epsilon \mathbb{Z} A_1) \dots \exp(\epsilon \mathbb{Z} A_{n_2}), \mathcal{O}_\epsilon = \sum_{k=1}^{n_2} \left(-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right) A_k$ and $F = \sum_{k=1}^{n_2} \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right] A_k$
$\mathbf{f}_{r,\epsilon} = r \times \mathbf{1}_{\Omega_\epsilon}, \mathbf{T}^\epsilon = \left\{ \begin{array}{l} \mathcal{T} : \mathcal{T} \text{ is a full-rank lattice of } \mathbf{P}_{\mathbf{J}(\lambda)}^*(\mathfrak{p}^*) \text{ and} \\ \sum_{\kappa \in \mathcal{T}} \mathbf{1}_{\beta_{\mathbf{J}(\lambda)}(F_\epsilon)}(\xi + \kappa) \leq 1 \text{ for every } \xi \in \mathbf{P}_{\mathbf{J}(\lambda)}^*(\mathfrak{p}^*) \end{array} \right\}$
$\left \det \left(\frac{id - e^{-\text{ad}(A)}}{\text{ad}(A)} \right) \right dA = w(\nu(A)) d(\nu(A)) = \rho(A) dA$
$\Gamma_P^\epsilon = \exp(\mathbb{Z}\text{-span}\{\mathcal{L}^\epsilon X_j : j \in \mathbf{J}(\lambda)\}), \Lambda_\epsilon^* = \mathbb{Z}\text{-span}\{(\mathcal{L}^\epsilon)^\top X_j^* : j \in \mathbf{J}(\lambda)\} \subset \mathfrak{p}^*$
$\Gamma^\epsilon = (\Gamma_M^\epsilon)^{-1} \Gamma_P^\epsilon = \exp(\epsilon \mathbb{Z} A_{n_1}) \dots \exp(\epsilon \mathbb{Z} A_1) \exp \left(\sum_{j \in \mathbf{J}(\lambda)} \mathbb{Z} \mathcal{L}^\epsilon X_j \right), \Theta_\lambda(\xi) = \left \det \text{Jac} [\beta_{\mathbf{J}(\lambda)} _{\mathcal{O}}]^{-1} \right $

and

Fix $\epsilon \in \mathbf{L}$, $\Omega_\epsilon^\circ = \exp \left(\left(-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right) A_1 \right) \dots \exp \left(\left(-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right) A_{n_2} \right)$
$\mathcal{O}_\epsilon = \left(-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right) A_1 + \dots + \left(-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right) A_{n_2} \subset \mathfrak{m}$,
$\Phi_{\mathbf{J}(\lambda)}^\epsilon(\mathbf{e}(a_1 A_1 + \dots + a_{n_2} A_{n_2})) = \beta_{\mathbf{J}(\lambda)}(a_1 A_1 + \dots + a_{n_2} A_{n_2})$
$\mathbf{s} \in C_c^\infty(M)$ and $\text{supp} \left(\mathbf{s} \circ \left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1} \right)$ is compact and contained in $\beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon)$
$\Lambda_\epsilon = \sum_{j \in \mathbf{J}(\lambda)} \mathbb{Z} \mathcal{L}^\epsilon X_j$ and $\Lambda_\epsilon^* = \sum_{j \in \mathbf{J}(\lambda)} \mathbb{Z} (\mathcal{L}^\epsilon)^\top X_j^*$
$\Gamma_P^\epsilon = \exp(\Lambda_\epsilon) = \exp \left(\sum_{j \in \mathbf{J}(\lambda)} \mathbb{Z} \mathcal{L}^\epsilon X_j \right), \Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi) d\xi = d\mu_M \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right)$
$\Upsilon(m) = \sqrt{\Psi_{\mathbf{J}(\lambda)}^\epsilon(\Phi_{\mathbf{J}(\lambda)}^\epsilon(m))} \left \det (\mathcal{L}^\epsilon)^\top \right \mathbf{s}(m)$
$\Gamma_M^\epsilon(\mathbf{s}) = \Gamma_M^\epsilon \subset M$, with $\inf \left\{ \sum_{\gamma \in \Gamma_M^\epsilon} \Upsilon(\gamma m) ^2 : m \in M \right\} > 0$, $\sup \left\{ \sum_{\gamma \in \Gamma_M^\epsilon} \Upsilon(\gamma m) ^2 : m \in M \right\} < \infty$

We recall that G is a simply connected, connected semi-direct product group of the type $G = P \rtimes M$ where P, M are closed subgroups of G . Moreover, it is assumed that there exists a linear functional λ in \mathfrak{p}^* such that the induced representation $\pi_\lambda = \text{ind}_P^G(\chi_\lambda)$ which is realized as acting in $L^2(M, d\mu_M)$ is an irreducible representation of G . Next, appealing to Pukansky's condition, we know that π_λ is irreducible if and only if

1. $\dim \mathfrak{p} = \dim \mathfrak{g} - \frac{1}{2} \dim (\mathbf{O}_\lambda)$
2. and the linear variety $\lambda + \mathfrak{p}^\perp$ is contained in \mathbf{O}_λ (the coadjoint orbit of λ .)

Definition 4 *A polarization algebra (or an ideal) in \mathfrak{g} subordinated to a linear functional λ is a subalgebra (or an ideal) \mathfrak{p} of \mathfrak{g} such that \mathfrak{p} is subordinated to λ and*

$$\dim \mathfrak{p} = \dim \mathfrak{g} - \frac{\dim (\mathbf{O}_\lambda)}{2}.$$

Lemma 5 *If $\pi_\lambda = \text{ind}_P^G(\chi_\lambda)$ is irreducible then \mathfrak{p} is a polarization algebra which is an ideal subordinated to the linear functional λ .*

Proof. Let us assume that π_λ is irreducible. Appealing to Result (32) on Page 38, [25]) together with the fact that P is assumed to be normal, it must be the case that $\mathfrak{p} = \log(P)$ is a polarization ideal for the linear functional λ . ■

Lemma 6 ([25] Page 43) *Assume that \mathfrak{p} is a polarization ideal subordinated to the linear functional λ . Then the following are equivalent.*

1. $\lambda + \mathfrak{p}^\perp \subset \mathbf{O}_\lambda$,
2. $\text{Ad}^*P(\lambda) = \lambda + \mathfrak{p}^\perp = \lambda + \mathfrak{m}^*$,
3. \mathfrak{p} is a polarization ideal for all linear functionals $\lambda + \lambda'$ where $\lambda' \in \mathfrak{m}^*$.

Lemma 7 *The irreducibility of π_λ implies that M acts freely on the linear functional λ .*

Proof. Since π_λ is irreducible, then $\lambda + \mathfrak{p}^\perp \subset \mathbf{O}_\lambda$. Next,

$$\dim \mathfrak{p} = \dim \mathfrak{g} - \frac{\dim (\mathbf{O}_\lambda)}{2}.$$

Note that

$$\begin{aligned} \dim \mathfrak{p} = \dim \mathfrak{g} - \frac{\dim (\mathbf{O}_\lambda)}{2} &\Rightarrow -\frac{\dim (\mathbf{O}_\lambda)}{2} = \dim \mathfrak{p} - \dim \mathfrak{g} \\ &\Rightarrow \frac{1}{2} \dim (\mathbf{O}_\lambda) = \dim \mathfrak{g} - \dim \mathfrak{p} \\ &\Rightarrow \frac{1}{2} \dim (\mathbf{O}_\lambda) = \dim \mathfrak{m} \\ &\Rightarrow \dim \mathbf{O}_\lambda = 2 \dim \mathfrak{m}. \end{aligned}$$

Thus, the coadjoint orbit of λ is a $2 \times \dim \mathfrak{m}$ -dimensional manifold. Since $\dim (\text{Ad}^*P(\lambda)) = \dim \mathfrak{m}$ it is now clear that

$$\dim (\text{Ad}^*M(\lambda)) = \dim (\mathfrak{m}) = \dim (\theta_\lambda(\mathfrak{m})).$$

Next, since M is a completely solvable Lie group, there exists a basis for \mathfrak{p} such that a matrix representation of $Ad^*(m)$ (which we denote by $[Ad^*(m)]$) is a lower triangular matrix with only real eigenvalues. As such,

$$\left\{ a \in \mathbb{R}^{n_2} : \left(\left(e^{-\text{ad}(a_{n_2}A_{n_2})} \dots e^{-\text{ad}(a_2A_2)} e^{-\text{ad}(a_1A_1)} \right) \Big|_{\mathfrak{p}} \right)^* \lambda = \lambda \right\}$$

cannot be a non-trivial subgroup discrete subgroup (or a lattice) of \mathbb{R}^{n_2} . Finally, since

$$\dim(Ad^*P(\lambda)) = \dim(\mathfrak{m})$$

it must then be the case that M acts freely on λ . ■

Define the map $\Phi_\lambda : M \rightarrow Ad(M)^* \lambda \subset \mathfrak{p}^*$ such that

$$\Phi_\lambda(m) = Ad(m)^* \lambda.$$

Lemma 8 Φ_λ is a smooth map that has constant rank. Moreover, $m \mapsto \Phi_\lambda(m)$ is an equivariant diffeomorphism and the rank of Φ_λ is equal to the dimension of M .

Proof. For any given $n \in M$, we have

$$\Phi_\lambda(nm) = Ad(nm)^* \lambda = Ad(n)^* Ad(m)^* \lambda = Ad(n)^* \Phi_\lambda(m).$$

Thus, the map Φ_λ is equivariant with respect to the multiplication action of M on itself and the coadjoint action of M on $Ad(M)^* \lambda$. Additionally, since the multiplicative action of M on itself is transitive, according to the Equivariant Rank Theorem (see [23], Theorem 7.5), Φ_λ is a smooth map that has constant rank. Next, appealing to [23], Theorem 7.19, since the isotropy group of λ is trivial by assumption, the map $m \mapsto \Phi_\lambda(m)$ is an equivariant diffeomorphism and the rank of this map must be equal to the dimension of M . ■

Next, we define the map $\mathbf{e} : \mathfrak{m} \rightarrow M$ by

$$\mathbf{e} \left(\sum_{k=1}^{n_2} a_k A_k \right) = \exp(a_1 A_1) \cdots \exp(a_{n_2} A_{n_2}) = \exp \left(\nu \left(\sum_{k=1}^{n_2} a_k A_k \right) \right) \quad (3.1)$$

Then \mathbf{e} is bijective and bi-analytic. Next, its inverse defines a global coordinate system on G (see Chapter 1, [25])

Lemma 9 $\theta_\lambda = \Phi_\lambda \circ \mathbf{e}$.

Proof. Given $a_1, \dots, a_{n_2} \in \mathbb{R}$,

$$\begin{aligned} \Phi_\lambda \left(\mathbf{e} \left(\sum_{k=1}^{n_2} a_k A_k \right) \right) &= \Phi_\lambda (\exp(a_1 A_1) \cdots \exp(a_{n_2} A_{n_2})) \\ &= \mathbf{C}(a_1, \dots, a_{n_2})^* \lambda = \theta_\lambda \left(\sum_{k=1}^{n_2} a_k A_k \right). \end{aligned}$$

■

Lemma 10 θ_λ defines an immersion of \mathfrak{m} into \mathfrak{p}^* of constant rank $\dim \mathfrak{m}$.

Proof. Since $\theta_\lambda = \Phi_\lambda \circ \mathbf{e}$, it follows that the rank of the smooth map θ_λ is equal to $\dim \mathfrak{m}$ at every $A \in \mathfrak{m}$. ■

Next, we recall that $D_{\theta_\lambda} = \text{Jac}_{\theta_\lambda}(0)$ is the Jacobian of the map θ_λ at the zero element in \mathfrak{m} . Let

$$\mathfrak{B}_{\mathfrak{p}} = (X_1, \dots, X_{n_1})$$

be a strong Malcev basis for \mathfrak{p} . Fix

$$\mathbf{J}(\lambda) = (j_1, \dots, j_{n_2}) \in \mathcal{T} \quad (3.2)$$

such that $D_{\theta_\lambda}(j_1, \dots, j_{n_2}) \in \mathcal{A}$. Define the map $\mathbf{P}_{\mathbf{J}(\lambda)} : \mathfrak{p} \rightarrow \mathfrak{p}$ such that

$$\mathbf{P}_{\mathbf{J}(\lambda)}(X_k) = \begin{cases} X_k & \text{if } k \in \mathbf{J}(\lambda) \\ 0 & \text{if } k \notin \mathbf{J}(\lambda) \end{cases}.$$

Put $a = (a_1, \dots, a_{n_2})$, and

$$A(a) = A = \sum_{k=1}^{n_2} a_k A_k.$$

From the definition of $\mathbf{P}_{\mathbf{J}(\lambda)}$, it is not hard to verify that $\mathbf{P}_{\mathbf{J}(\lambda)}$ is a linear map. Moreover, $\mathbf{P}_{\mathbf{J}(\lambda)}^2 = \mathbf{P}_{\mathbf{J}(\lambda)}$ and the nullspace and range of $\mathbf{P}_{\mathbf{J}(\lambda)}$ are orthogonal to each other with respect to the natural dot product on \mathfrak{p} . Therefore, $\mathbf{P}_{\mathbf{J}(\lambda)}$ defines an orthogonal projection of rank n_2 on \mathfrak{p} . $\mathbf{P}_{\mathbf{J}(\lambda)}$ is an orthogonal projection of rank n_2 on the Lie algebra \mathfrak{p} .

Lemma 11 Let X be an element of the range of $\mathbf{P}_{\mathbf{J}(\lambda)}$. Then given $(a_1, \dots, a_{n_2}) \in \mathbb{R}^{n_2}$,

$$\langle \mathbf{C}(a_1, \dots, a_{n_2})^* \lambda, X \rangle = \langle \mathbf{P}_{\mathbf{J}(\lambda)}^* \mathbf{C}(a_1, \dots, a_{n_2})^* \lambda, X \rangle.$$

Proof. If X belongs to the range of $\mathbf{P}_{\mathbf{J}(\lambda)}$ then $X = \mathbf{P}_{\mathbf{J}(\lambda)} X'$ for some element X' in the Lie algebra \mathfrak{p} . Consequently,

$$\begin{aligned} \langle \mathbf{C}(a_1, \dots, a_{n_2})^* \lambda, X \rangle &= \langle \mathbf{C}(a_1, \dots, a_{n_2})^* \lambda, \mathbf{P}_{\mathbf{J}(\lambda)} X' \rangle \\ &= \langle \mathbf{C}(a_1, \dots, a_{n_2})^* \lambda, \mathbf{P}_{\mathbf{J}(\lambda)} \mathbf{P}_{\mathbf{J}(\lambda)} X' \rangle \\ &= \langle \mathbf{P}_{\mathbf{J}(\lambda)}^* \mathbf{C}(a_1, \dots, a_{n_2})^* \lambda, \mathbf{P}_{\mathbf{J}(\lambda)} X' \rangle \\ &= \langle \mathbf{P}_{\mathbf{J}(\lambda)}^* \mathbf{C}(a_1, \dots, a_{n_2})^* \lambda, X \rangle. \end{aligned}$$

■

Appealing to the preceding lemma, it is clear that if X is an element of the range of $\mathbf{P}_{\mathbf{J}(\lambda)}$ then

$$[\pi_\lambda(\exp X) \mathbf{f}](\mathbf{e}(A)) = e^{2\pi i \langle \mathbf{P}_{\mathbf{J}(\lambda)}^* \mathbf{C}(a)^* \lambda, X \rangle} \mathbf{f}(\mathbf{e}(A)).$$

Next, let

$$\beta_{\mathbf{J}(\lambda)} : \mathfrak{m} \rightarrow \sum_{k \in \mathbf{J}(\lambda)} \mathbb{R} X_k^* = \mathbf{P}_{\mathbf{J}(\lambda)}^* (\mathfrak{p}^*) \quad (3.3)$$

such that

$$\beta_{\mathbf{J}(\lambda)}(A) = \mathbf{P}_{\mathbf{J}(\lambda)}^* \mathbf{C}(a)^* \lambda = \mathbf{P}_{\mathbf{J}(\lambda)}^* \theta_\lambda(A).$$

Lemma 12 *There exists an open set \mathcal{O} around the neutral element in \mathfrak{m} such that the restriction of $\beta_{\mathbf{J}(\lambda)}$ to \mathcal{O} defines a diffeomorphism between \mathcal{O} and $\beta_{\mathbf{J}(\lambda)}(\mathcal{O})$. Moreover, there exists $\epsilon > 0$ such that given*

$$\mathcal{O}_\epsilon = \left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) A_1 + \cdots + \left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) A_{n_2} \subset \mathfrak{m} \quad (3.4)$$

the restriction of $\beta_{\mathbf{J}(\lambda)}$ to \mathcal{O}_ϵ determines a diffeomorphism between \mathcal{O}_ϵ and $\beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon)$.

Proof. Since the Jacobian of the map $\beta_{\mathbf{J}(\lambda)}$ at zero is invertible, there exists a neighborhood of the zero element such that the restriction of $\beta_{\mathbf{J}(\lambda)}$ to such a neighborhood has constant rank. Thus, appealing to Proposition 5.16, [23], there exists an open set \mathcal{O} around the neutral element in \mathfrak{m} such that the restriction of $\beta_{\mathbf{J}(\lambda)}$ to \mathcal{O} defines a diffeomorphism between \mathcal{O} and its range $\beta_{\mathbf{J}(\lambda)}(\mathcal{O})$. The second part of the lemma is trivial and we shall omit its proof. ■

Proposition 13 *Let $\epsilon > 0$ and set*

$$\Omega_\epsilon = \exp\left(\left[-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) A_1\right) \exp\left(\left[-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) A_2\right) \cdots \exp\left(\left[\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) A_{n_2}\right).$$

Then

$$\{\gamma^{-1}\Omega_\epsilon : \gamma \in \exp(\epsilon\mathbb{Z}A_1) \exp(\epsilon\mathbb{Z}A_2) \cdots \exp(\epsilon\mathbb{Z}A_{n_2})\}$$

is a measurable partition of M .

Proof. We shall prove this claim by induction on the dimension of M . Clearly for $n_2 = 1$ the base case holds. Next, let

$$M_1 = \exp(\mathbb{R}A_1) \cdots \exp(\mathbb{R}A_{n_2-1}) \text{ and } H_1 = \exp(\mathbb{R}A_{n_2})$$

Put $M = M_1 H_1$,

$$\Omega_1 = \exp\left[-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) A_1 \cdots \exp\left[-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) A_{n_2-1}$$

and $\Gamma_M^1 = \exp(\epsilon\mathbb{Z}A_1) \cdots \exp(\epsilon\mathbb{Z}A_{n_2-1})$. Next define

$$\Omega = \exp\left[-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) A_1 \cdots \exp\left[-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) A_{n_2}$$

and $\Gamma_M = \exp(\epsilon\mathbb{Z}A_1) \cdots \exp(\epsilon\mathbb{Z}A_{n_2})$. Let $m \in M$. Then, there exists a unique pair $(m_{n_2-1}, m_{n_2}) \in M_1 \times H_1$ such that $m = m_{n_2-1} m_{n_2}$. Appealing to the inductive hypothesis, $m = (\gamma_{n_2-1}^{-1} \omega_{n_2-1}) (\gamma_{n_2}^{-1} \omega_{n_2})$ where

$$\gamma_{n_2-1} \in \Gamma_M^1, \omega_{n_2-1} \in \Omega_1, \gamma_{n_2} \in \exp(\epsilon\mathbb{Z}A_{n_2}), \omega_{n_2} \in H_1.$$

Next, let e be the neutral element in Γ_M . Then

$$\begin{aligned} m &= e (\gamma_{n_2-1}^{-1} \omega_{n_2-1}) (\gamma_{n_2}^{-1} \omega_{n_2}) \\ &= (\gamma_{n_2}^{-1} \gamma_{n_2}) (\gamma_{n_2-1}^{-1} \omega_{n_2-1}) (\gamma_{n_2}^{-1} \omega_{n_2}) \\ &= \gamma_{n_2}^{-1} (\gamma_{n_2} (\gamma_{n_2-1}^{-1} \omega_{n_2-1}) \gamma_{n_2}^{-1}) \omega_{n_2}. \end{aligned}$$

Now, since M_1 is a normal subgroup of M , $\gamma_{n_2-1}^{-1} \omega_{n_2-1} \in M_1$ it follows that $\gamma_{n_2} (\gamma_{n_2-1}^{-1} \omega_{n_2-1}) \gamma_{n_2}^{-1} \in M_1$. Thus, there exists a unique pair $(\tilde{\gamma}_{n_2-1}, \tilde{\omega}_{n_2-1}) \in \Gamma_M^1 \times \Omega_1$ such that

$$\gamma_{n_2} (\gamma_{n_2-1}^{-1} \omega_{n_2-1}) \gamma_{n_2}^{-1} = \tilde{\gamma}_{n_2-1}^{-1} \tilde{\omega}_{n_2-1}.$$

Consequently,

$$\begin{aligned} m &= \gamma_{n_2}^{-1} \underbrace{(\gamma_{n_2} (\gamma_{n_2-1}^{-1} \omega_{n_2-1}) \gamma_{n_2}^{-1})}_{=\tilde{\gamma}_{n_2-1}^{-1} \tilde{\omega}_{n_2-1}} \omega_{n_2} \\ &= \gamma_{n_2}^{-1} (\tilde{\gamma}_{n_2-1}^{-1} \tilde{\omega}_{n_2-1}) \omega_{n_2} \\ &= (\gamma_{n_2}^{-1} \tilde{\gamma}_{n_2-1}^{-1}) (\tilde{\omega}_{n_2-1} \omega_{n_2}) \\ &= (\tilde{\gamma}_{n_2-1} \gamma_{n_2})^{-1} (\tilde{\omega}_{n_2-1} \omega_{n_2}). \end{aligned}$$

Since the factorization above is unique,

$$\{\gamma^{-1} \Omega_\epsilon : \gamma \in \exp(\epsilon \mathbb{Z} A_1) \exp(\epsilon \mathbb{Z} A_2) \cdots \exp(\epsilon \mathbb{Z} A_{n_2})\}$$

forms a measurable partition of M . ■

Fix $\epsilon \in \mathbf{L}$,

$$\mathbf{L} = \left\{ s > 0 : \sum_{k=1}^{n_2} \left[-\frac{s}{2}, \frac{s}{2} \right] A_k \subset \mathcal{O} \right\} \subset \mathbb{R}. \quad (3.5)$$

Next, we define

$$\Omega_\epsilon = \exp \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right) A_1 \exp \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right) A_2 \cdots \exp \left[\frac{\epsilon}{2}, \frac{\epsilon}{2} \right) A_{n_2} \quad (3.6)$$

such that

$$\mathcal{O}_\epsilon = \left(-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right) A_1 + \cdots + \left(-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right) A_{n_2} \subset \mathfrak{m} \quad (3.7)$$

is an open set around the neutral element in \mathfrak{m} such that the restriction of $\beta_{\mathbf{J}(\lambda)}$ to \mathcal{O}_ϵ defines a diffeomorphism between \mathcal{O}_ϵ and its image set $\beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon)$. Fix a discrete set $\Gamma_M \subset M$,

$$\Gamma_M = \Gamma_M^\epsilon = \exp(\epsilon \mathbb{Z} A_1) \exp(\epsilon \mathbb{Z} A_2) \cdots \exp(\epsilon \mathbb{Z} A_{n_2})$$

such that

$$\{\gamma^{-1} \Omega_\epsilon : \gamma \in \Gamma_M\}$$

is a $d\mu_M$ -measurable partition of M . Next, put

$$F_\epsilon = \sum_{k=1}^{n_2} \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right) A_k \subset \mathfrak{m}$$

such that $\mathbf{e}(F_\epsilon) = \Omega_\epsilon$. Fix $\mathbf{J}(\lambda) = (j_1 < \cdots < j_m)$ such that $D_{\theta_\lambda}(\mathbf{J}(\lambda)) \in \mathcal{A}$ where

$$\mathcal{A} = \{D_{\theta_\lambda}(\mathbf{I}) : \mathbf{I} \in \mathcal{T} \text{ and } \det(D_{\theta_\lambda}(\mathbf{I})) \neq 0\}.$$

Finally, define $V_{\mathbf{J}(\lambda)} = \mathbb{R}\text{-span}\{X_j : j \in \mathbf{J}(\lambda)\}$ and

$$\mathbf{T}^\epsilon = \left\{ \begin{array}{l} \mathcal{T} : \mathcal{T} \text{ is a full-rank lattice of } \mathbf{P}_{\mathbf{J}(\lambda)}^*(\mathbf{p}^*) \text{ and} \\ \sum_{\kappa \in \mathcal{T}} \mathbf{1}_{\beta_{\mathbf{J}(\lambda)}(F_\epsilon)}(\xi + \kappa) \leq 1 \text{ for every } \xi \in \mathbf{P}_{\mathbf{J}(\lambda)}^*(\mathbf{p}^*) \end{array} \right\}.$$

Moreover, for a linear operator $\mathcal{L}^\epsilon : V_{\mathbf{J}(\lambda)} \rightarrow V_{\mathbf{J}(\lambda)}$ we define $\Lambda_\epsilon(\mathcal{L}^\epsilon) = \mathbb{Z}\text{-span}\{\mathcal{L}^\epsilon X_j : j \in \mathbf{J}(\lambda)\}$ and

$$\Lambda_\epsilon^*(\mathcal{L}^\epsilon) = \mathbb{Z}\text{-span} \left\{ (\mathcal{L}^\epsilon)^\top X_j^* : j \in \mathbf{J}(\lambda) \right\}.$$

Lemma 14 \mathbf{T}^ϵ is a non-empty set. In other words, there exists an invertible linear operator $\mathcal{L}^\epsilon : V_{\mathbf{J}(\lambda)} \rightarrow V_{\mathbf{J}(\lambda)}$ such that $\beta_{\mathbf{J}(\lambda)}(F_\epsilon)$ is contained in a measurable fundamental domain of a dual lattice $\Lambda_\epsilon^*(\mathcal{L}^\epsilon)$.

Proof. First, we claim that $\beta_{\mathbf{J}(\lambda)}(F_\epsilon)$ is a bounded set. Indeed, since the compact set

$$\sum_{k=1}^{n_2} \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right] A_k$$

is properly contained in \mathcal{O} and because the restriction of $\beta_{\mathbf{J}(\lambda)}$ to \mathcal{O} is a diffeomorphism, it follows that the image of $\sum_{k=1}^{n_2} \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right] A_k$ under $\beta_{\mathbf{J}(\lambda)}$ is compact as well. Moreover, the fact that $\beta_{\mathbf{J}(\lambda)}(F_\epsilon)$ is a subset of

$$\beta_{\mathbf{J}(\lambda)} \left(\sum_{k=1}^{n_2} \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right] A_k \right)$$

implies that $\beta_{\mathbf{J}(\lambda)}(F_\epsilon)$ is a bounded set. Next, let

$$\delta_\epsilon = \sup \left\{ \left\| \beta_{\mathbf{J}(\lambda)} \left(\sum_{k=1}^{n_2} a_k A_k \right) \right\|_{\max} : (a_1, \dots, a_{n_2}) \in \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right]^{n_2} \right\} > 0. \quad (3.8)$$

Secondly, put

$$\mathfrak{F} = \sum_{j \in \mathbf{J}(\lambda)} [-\delta_\epsilon, \delta_\epsilon] X_j^*.$$

Note that \mathfrak{F} is a fundamental domain for the lattice $\sum_{j \in \mathbf{J}(\lambda)} 2\delta_\epsilon \mathbb{Z} X_j^*$. Thirdly, from the definition of δ_ϵ , it is clear that $\beta_{\mathbf{J}(\lambda)}(F_\epsilon) \subseteq \mathfrak{F}$. Define the linear map $\mathcal{L}^\epsilon : V_{\mathbf{J}(\lambda)} \rightarrow V_{\mathbf{J}(\lambda)}$ such that

$$\mathcal{L}^\epsilon \left(\sum_{j \in \mathbf{J}(\lambda)} x_j X_j \right) = \sum_{j \in \mathbf{J}(\lambda)} \frac{x_j}{2\delta_\epsilon} X_j.$$

In other words, the matrix representation of \mathcal{L}^ϵ with respect to the ordered basis $(X_j)_{j \in \mathbf{J}(\lambda)}$ is given by

$$[\mathcal{L}^\epsilon]_{(X_j)_{j \in \mathbf{J}(\lambda)}} = \begin{bmatrix} \frac{1}{2\delta_\epsilon} & & \\ & \ddots & \\ & & \frac{1}{2\delta_\epsilon} \end{bmatrix}, \text{ and } [(\mathcal{L}^\epsilon)^\top]_{(X_j^*)_{j \in \mathbf{J}(\lambda)}} = \begin{bmatrix} 2\delta_\epsilon & & \\ & \ddots & \\ & & 2\delta_\epsilon \end{bmatrix}.$$

Thus, for every $\xi \in \mathbf{P}_{\mathbf{J}(\lambda)}^* (\mathfrak{p}^*)$,

$$\sum_{\kappa \in \Lambda_{\epsilon}^* (\mathcal{L}^{\epsilon})} \mathbf{1}_{\beta_{\mathbf{J}(\lambda)}(F_{\epsilon})} (\xi + \kappa) \leq 1.$$

■

Example 15 *Let G be a connected, simply connected solvable Lie group with Lie algebra \mathfrak{g} spanned by $\{X_1, X_2, X_3, A_1, A_2\}$ with non-trivial Lie brackets*

$$[A_1, X_2] = X_1, [A_2, X_1] = X_1, [A_2, X_2] = X_2, [A_2, X_3] = -2X_3$$

and

$$[\text{ad} X_1]_{(X_1, X_2, X_3)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [\text{ad} X_2]_{(X_1, X_2, X_3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Fix $\lambda = X_1^* + X_2^* + X_3^*$. With straightforward computations, we obtain

$$\beta_{\mathbf{J}(\lambda)}(t_1, t_2) = \begin{cases} (e^{-t_2}, e^{-t_2}(t_1 - 1)) & \text{if } \mathbf{J}(\lambda) = (1, 2) \\ (e^{-t_2}(t_1 - 1), e^{2t_2}) & \text{if } \mathbf{J}(\lambda) = (2, 3) \end{cases}.$$

Finally, the orbital data corresponding to the linear functional λ is given by $\{\beta_{(1,2)}, \beta_{(2,3)}\}$. Next, fix the map

$$\beta_{(2,3)}(t) = (e^{-t_2}(t_1 - 1), e^{2t_2})$$

in the orbital data corresponding to λ . Clearly $\beta_{(2,3)}$ defines a diffeomorphism between \mathbb{R}^2 and $\beta_{(2,3)}(\mathbb{R}^2)$. It follows that

$$\|\beta_{(2,3)}(t)\|_{\max} = \max \left\{ e^{2t_2}, \frac{|t_1 - 1|}{e^{t_2}} \right\}$$

and

$$\delta_{\epsilon} = \sup \left\{ \max \left\{ e^{2t_2}, \frac{|t_1 - 1|}{e^{t_2}} \right\} : t \in \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right]^2 \right\}.$$

Thus, $\beta_{(2,3)} \left(\left[-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right]^2 \right)$ is contained in a fundamental domain of $2\delta_{\epsilon}\mathbb{Z}X_2^* + 2\delta_{\epsilon}\mathbb{Z}X_3^*$.

4 Proofs of main theorems

4.1 Proof of Theorem 2

Fix $\epsilon \in \mathbf{L}$ and

$$\Omega_{\epsilon} = \exp \left(\left[-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right] A_1 \right) \exp \left(\left[-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right] A_2 \right) \cdots \exp \left(\left[\frac{\epsilon}{2}, \frac{\epsilon}{2} \right] A_{n_2} \right).$$

Fix $\mathcal{L}^{\epsilon} \in \mathbf{T}^{\epsilon}$ and put $\Gamma_P^{\epsilon} = \exp(\Lambda_{\epsilon})$. For any given positive function r defined on Ω_{ϵ} , let

$$\mathbf{f}_{r,\epsilon} = r \times \mathbf{1}_{\Omega_{\epsilon}}$$

and let

$$b(\xi) = [\beta_{\mathbf{J}(\lambda)}]^{-1}(\xi).$$

Recall that ρ is the Radon-Nikodym derivative given by

$$d\mu_M \left(\mathbf{e} \left(\sum_{k=1}^{n_2} a_k A_k \right) \right) = d\mu_M (\exp(a_1 A_1) \cdots \exp(a_{n_2} A_{n_2})) = \rho(A) dA \quad (4.1)$$

where $d\mu_M$ is a left Haar measure on the solvable group M and dA is the Lebesgue measure on $\mathfrak{m} = \mathbb{R}^{n_2}$.

Lemma 16 *Let $\mathbf{h} \in L^2(\Omega_\epsilon, d\mu_M) \cap C(\Omega_\epsilon)$. If*

$$\xi \mapsto r(\mathbf{e}(b(\xi))) \rho(b(\xi)) \Theta_\lambda(\xi)$$

is square-integrable over $\beta_{\mathbf{J}(\lambda)}(F_\epsilon)$ with respect to the Lebesgue measure then

$$\begin{aligned} & \sum_{\alpha \in \Gamma_P^\epsilon} \left| \langle \mathbf{h}, \pi_\lambda(\alpha) \mathbf{f}_{r,\epsilon} \rangle_{L^2(\Omega_\epsilon, d\mu_M)} \right|^2 \\ &= \left| \det(\mathcal{L}^\epsilon)^\top \right| \int_{F_\epsilon} |\mathbf{h}(\mathbf{e}(A))|^2 [|r(\mathbf{e}(A))|^2 \rho(A)^2 \Theta_\lambda(\beta_{\mathbf{J}(\lambda)}(A))] dA. \end{aligned}$$

Proof. Let $\mathbf{f}_{r,\epsilon}, \mathbf{h}$ be as defined in the statement of the lemma. Then

$$\begin{aligned} & \sum_{\alpha \in \Gamma_P^\epsilon} \left| \langle \mathbf{h}, \pi_\lambda(\alpha) \mathbf{f}_{r,\epsilon} \rangle_{L^2(\Omega_\epsilon, d\mu_M)} \right|^2 \\ &= \sum_{\exp(X) \in \Gamma_P^\epsilon} \left| \langle \mathbf{h}, \pi_\lambda(\exp X) \mathbf{f}_{r,\epsilon} \rangle_{L^2(\Omega_\epsilon, d\mu_M)} \right|^2 \\ &= \sum_{\exp(X) \in \Gamma_P^\epsilon} \left| \int_M \mathbf{h}(m) e^{-2\pi i \langle \lambda, \log(m^{-1} \exp(X)m) \rangle} \overline{\mathbf{f}_{r,\epsilon}(m)} d\mu_M(m) \right|^2. \end{aligned}$$

Since the support of $\mathbf{f}_{r,\epsilon}$ is equal to Ω_ϵ , it follows that

$$\begin{aligned} & \sum_{\alpha \in \Gamma_P^\epsilon} \left| \langle \mathbf{h}, \pi_\lambda(\alpha) \mathbf{f}_{r,\epsilon} \rangle_{L^2(\Omega_\epsilon, d\mu_M)} \right|^2 \\ &= \sum_{\exp(X) \in \Gamma_P^\epsilon} \left| \int_{\Omega_\epsilon} \mathbf{h}(m) e^{-2\pi i \langle \lambda, \log(m^{-1} \exp(X)m) \rangle} \overline{\mathbf{f}_{r,\epsilon}(m)} d\mu_M(m) \right|^2. \end{aligned}$$

Next, since

$$\mathbf{e}(F_\epsilon) = \Omega_\epsilon,$$

for a given $m \in \Omega_\epsilon$, there exists a unique $A \in F_\epsilon$ such that

$$m = \mathbf{e}(A)$$

and

$$\begin{aligned}
& \sum_{\alpha \in \Gamma_P^\epsilon} \left| \langle \mathbf{h}, \pi_\lambda(\alpha) \mathbf{f}_{r,\epsilon} \rangle_{L^2(\Omega_\epsilon, d\mu_M)} \right|^2 \\
&= \sum_{\exp(X) \in \Gamma_P^\epsilon} \left| \int_{F_\epsilon} \mathbf{h}(\mathbf{e}(A)) e^{-2\pi i \langle D_{J(\lambda)} \mathbf{C}^{(a)*\lambda, X} \rangle} \overline{\mathbf{f}_{r,\epsilon}(\mathbf{e}(A))} \underbrace{d\mu_M(\mathbf{e}(A))}_{=\rho(A)dA} \right|^2 \\
&= \sum_{\exp(X) \in \Gamma_P^\epsilon} \left| \int_{F_\epsilon} \mathbf{h}(\mathbf{e}(A)) e^{-2\pi i \langle D_{J(\lambda)} \mathbf{C}^{(a)*\lambda, X} \rangle} \overline{\mathbf{f}_{r,\epsilon}(\mathbf{e}(A))} \rho(A) dA \right|^2 \\
&= \sum_{\exp(X) \in \Gamma_P^\epsilon} \left| \int_{F_\epsilon} e^{-2\pi i \langle \beta_{J(\lambda)}(A), X \rangle} (r(\mathbf{e}(A)) \times \mathbf{h}(\mathbf{e}(A)) \rho(A)) dA \right|^2 = (*)
\end{aligned}$$

Set $\xi = \beta_{J(\lambda)}(A)$ and let

$$\Psi_{r, J(\lambda)}(\xi) = r(\mathbf{e}(b(\xi))) \mathbf{h}(\mathbf{e}(b(\xi))) \rho(b(\xi)).$$

The change of variable $A = b(\xi)$ gives

$$(*) = \sum_{\exp(X) \in \Gamma_P^\epsilon} \left| \int_{\beta_{J(\lambda)}(F_\epsilon)} e^{-2\pi i \langle \xi, X \rangle} \Psi_{r, J(\lambda)}(\xi) d(b(\xi)) \right|^2.$$

Recall that $\Theta_\lambda(\xi)$ is the Radon-Nikodym derivative given by

$$\frac{d(b(\xi))}{d\xi} = \Theta_\lambda(\xi).$$

Since $\beta_{J(\lambda)}(F_\epsilon)$ is contained in a fundamental domain of

$$\Lambda_\epsilon^*(\mathcal{L}^\epsilon) = \mathbb{Z}\text{-span} \left\{ (\mathcal{L}^\epsilon)^\top X_j^* : j \in \mathbf{J}(\lambda) \right\},$$

the system

$$\left\{ \frac{e^{-2\pi i \langle \xi, X \rangle} \times \mathbf{1}_{\beta_{J(\lambda)}(F_\epsilon)}(\xi)}{|\det(\mathcal{L}^\epsilon)^\top|^{1/2}} : X \in \Lambda_\epsilon \right\} \quad (4.2)$$

is a Parseval frame for $L^2(\beta_{J(\lambda)}(F_\epsilon))$. Appealing to Hölder's inequality,

$$\Psi_{r, J(\lambda)}(\xi) \Theta_\lambda(\xi) = \mathbf{h}(\mathbf{e}(b(\xi))) r(\mathbf{e}(b(\xi))) \times \rho(b(\xi)) \Theta_\lambda(\xi)$$

is an element of $L^1(\beta_{J(\lambda)}(F_\epsilon))$. Consequently,

$$\begin{aligned}
(*) &= \sum_{\exp(X) \in \Gamma_P^\epsilon} \left| \int_{\beta_{J(\lambda)}(F_\epsilon)} e^{-2\pi i \langle \xi, X \rangle} \Psi_{r, J(\lambda)}(\xi) \Theta_\lambda(\xi) d\xi \right|^2 \\
&= \sum_{\exp(X) \in \Gamma_P^\epsilon} \left| \int_{\beta_{J(\lambda)}(F_\epsilon)} |\det(\mathcal{L}^\epsilon)^\top|^{1/2} \left[\frac{e^{-2\pi i \langle \xi, X \rangle}}{|\det(\mathcal{L}^\epsilon)^\top|^{1/2}} \right] \Psi_{r, J(\lambda)}(\xi) \Theta_\lambda(\xi) d\xi \right|^2 = (**)
\end{aligned}$$

Next,

$$\begin{aligned}
(**) &= \sum_{\exp(X) \in \Gamma_{\mathcal{P}}^{\epsilon}} \left| \int_{\beta_{\mathbf{J}(\lambda)}(F_{\epsilon})} \left| \det(\mathcal{L}^{\epsilon})^{\top} \right|^{1/2} \left[\frac{e^{-2\pi i \langle \xi, X \rangle}}{\left| \det(\mathcal{L}^{\epsilon})^{\top} \right|^{1/2}} \right] r(\mathbf{e}(b(\xi))) \right. \\
&\quad \times \mathbf{h}(\mathbf{e}(b(\xi))) \rho(b(\xi)) \Theta_{\lambda}(\xi) d\xi \Big|^2 \\
&= \int_{\beta_{\mathbf{J}(\lambda)}(F_{\epsilon})} \left| \left| \det(\mathcal{L}^{\epsilon})^{\top} \right|^{1/2} r(\mathbf{e}(b(\xi))) \right|^2 |\mathbf{h}(\mathbf{e}(b(\xi))) \rho(b(\xi)) \Theta_{\lambda}(\xi)|^2 d\xi \\
&= (***)
\end{aligned}$$

and

$$(***) = \int_{\beta_{\mathbf{J}(\lambda)}(F_{\epsilon})} \left| \left| \det(\mathcal{L}^{\epsilon})^{\top} \right|^{1/2} \mathbf{h}(\mathbf{e}(b(\xi))) \right|^2 |r(\mathbf{e}(b(\xi))) \rho(b(\xi)) \Theta_{\lambda}(\xi)|^2 d\xi.$$

Finally, the change of variable

$$A = b(\xi) = [\beta_{\mathbf{J}(\lambda)}]^{-1}(\xi)$$

yields

$$\begin{aligned}
(***) &= \left| \det(\mathcal{L}^{\epsilon})^{\top} \right| \int_{F_{\epsilon}} |\mathbf{h}(\mathbf{e}(A))|^2 \left[|r(\mathbf{e}(A)) \rho(A) \Theta_{\lambda}(\beta_{\mathbf{J}(\lambda)}(A))|^2 \right] d(\beta_{\mathbf{J}(\lambda)}(A)) \\
&= \left| \det(\mathcal{L}^{\epsilon})^{\top} \right| \int_{F_{\epsilon}} |\mathbf{h}(\mathbf{e}(A))|^2 [|r(\mathbf{e}(A)) \rho(A) \Theta_{\lambda}(\beta_{\mathbf{J}(\lambda)}(A))|] \Theta_{\lambda}(\beta_{\mathbf{J}(\lambda)}(A)) d(\beta_{\mathbf{J}(\lambda)}(A)) \\
&= (****)
\end{aligned}$$

Since $\Theta_{\lambda}(\xi)$ is the absolute value of the determinant of the Jacobian of $[\beta_{\mathbf{J}(\lambda)}|_{\mathcal{O}}]^{-1}$,

$$d([\beta_{\mathbf{J}(\lambda)}|_{\mathcal{O}}]^{-1}(\xi)) = \Theta_{\lambda}(\xi) d\xi.$$

As such,

$$dA = d([\beta_{\mathbf{J}(\lambda)}|_{\mathcal{O}}]^{-1}(\beta_{\mathbf{J}(\lambda)}(A))) = \Theta_{\lambda}(\beta_{\mathbf{J}(\lambda)}(A)) d(\beta_{\mathbf{J}(\lambda)}(A)),$$

and

$$(***) = \left| \det(\mathcal{L}^{\epsilon})^{\top} \right| \int_{F_{\epsilon}} |\mathbf{h}(\mathbf{e}(A))|^2 [|r(\mathbf{e}(A))|^2 \rho(A)^2 \Theta_{\lambda}(\beta_{\mathbf{J}(\lambda)}(A))] dA.$$

■

Lemma 17 *The function $A \mapsto (\Theta_{\lambda}(\beta_{\mathbf{J}(\lambda)}(A)))^{-1}$ is integrable on every compact measurable subset of \mathcal{O} .*

Proof. We shall prove that if \mathbf{K} is a compact Lebesgue measurable subset of \mathcal{O} then

$$\int_{\mathbf{K}} \frac{1}{\Theta_{\lambda}(\beta_{\mathbf{J}(\lambda)}(A))} dA < \infty.$$

Letting $\xi = \beta_{\mathbf{J}(\lambda)}(A)$,

$$\begin{aligned} \int_{\mathbf{K}} \frac{1}{\Theta_{\lambda}(\beta_{\mathbf{J}(\lambda)}(A))} dA &= \int_{\beta_{\mathbf{J}(\lambda)}(\mathbf{K})} \frac{1}{\Theta_{\lambda}(\xi)} d(b(\xi)) \\ &= \int_{\beta_{\mathbf{J}(\lambda)}(\mathbf{K})} \frac{\Theta_{\lambda}(\xi)}{\Theta_{\lambda}(\xi)} d\xi \\ &= \int_{\beta_{\mathbf{J}(\lambda)}(\mathbf{K})} d\xi \\ &= |\beta_{\mathbf{J}(\lambda)}(\mathbf{K})|. \end{aligned}$$

Using the fact that $\beta_{\mathbf{J}(\lambda)}$ is a diffeomorphism between \mathcal{O} and $\beta_{\mathbf{J}(\lambda)}(\mathcal{O})$ together with the assumption that \mathbf{K} is a compact measurable subset of \mathcal{O} , it follows that $\beta_{\mathbf{J}(\lambda)}(\mathbf{K})$ is compact. Consequently,

$$\int_{\mathbf{K}} \frac{dA}{\Theta_{\lambda}(\beta_{\mathbf{J}(\lambda)}(A))} = |\beta_{\mathbf{J}(\lambda)}(\mathbf{K})| < \infty.$$

■

Lemma 18 *If $r(\mathbf{e}(A)) = (\rho(A) \Theta_{\lambda}(\beta_{\mathbf{J}(\lambda)}(A)))^{-1/2}$ then $\mathbf{f}_{r,\epsilon} = r \times \mathbf{1}_{\Omega_{\epsilon}}$ is square-integrable with respect to the Haar measure on M .*

Proof. We note that

$$\begin{aligned} |r(\mathbf{e}(A))|^2 \rho(A)^2 \Theta_{\lambda}(\beta_{\mathbf{J}(\lambda)}(A)) &= \rho(A) \\ \Leftrightarrow |r(\mathbf{e}(A))|^2 \Theta_{\lambda}(\beta_{\mathbf{J}(\lambda)}(A)) &= \frac{1}{\rho(A)} \\ \Leftrightarrow |r(\mathbf{e}(A))| &= \left(\frac{1}{\rho(A) \Theta_{\lambda}(\beta_{\mathbf{J}(\lambda)}(A))} \right)^{1/2}. \end{aligned}$$

Next, it is clear that $\mathbf{f}_{r,\epsilon} \in L^2(\Omega_{\epsilon}, d\mu_M)$ if and only if $\frac{1}{\theta_{\lambda}(\beta_{\mathbf{J}(\lambda)}(A))}$ is integrable over F_{ϵ} . Indeed,

$$\int_{F_{\epsilon}} |\mathbf{f}_{r,\epsilon}(\mathbf{e}(A))|^2 \rho(A) dA = \int_{F_{\epsilon}} \frac{dA}{\Theta_{\lambda}(\beta_{\mathbf{J}(\lambda)}(A))}.$$

Appealing to Lemma 17,

$$\int_{F_{\epsilon}} \frac{1}{\Theta_{\lambda}(\beta_{\mathbf{J}(\lambda)}(A))} dA < \infty.$$

■

Lemma 19 *Let $\mathbf{h} \in L^2(\Omega_{\epsilon}, d\mu_M) \cap C(\Omega_{\epsilon})$. If*

$$r(\mathbf{e}(A)) = \frac{1}{\sqrt{\rho(A) \Theta_{\lambda}(\beta_{\mathbf{J}(\lambda)}(A))}}$$

then

$$\sum_{\alpha \in \Gamma_P^{\epsilon}} \left| \langle \mathbf{h}, \pi_{\lambda}(\alpha) \mathbf{f}_{r,\epsilon} \rangle_{L^2(\Omega_{\epsilon}, d\mu_M)} \right|^2 = \left| \det(\mathcal{L}^{\epsilon})^{\top} \right| \times \|\mathbf{h}\|_{L^2(\Omega_{\epsilon}, d\mu_M)}^2.$$

Proof. First, if $r(\mathbf{e}(A)) = (\rho(A) \Theta_\lambda(\beta_{\mathbf{J}(\lambda)}(A)))^{-1/2}$ then

$$\begin{aligned} r(\mathbf{e}(b(\xi))) \rho(b(\xi)) \Theta_\lambda(\xi) &= \frac{\rho(b(\xi)) \Theta_\lambda(\xi)}{\left(\rho(b(\xi)) \Theta_\lambda\left(\beta_{\mathbf{J}(\lambda)}\left((\beta_{\mathbf{J}(\lambda)})^{-1}(\xi)\right)\right)\right)^{1/2}} \\ &= \frac{\rho(b(\xi)) \Theta_\lambda(\xi)}{\rho(b(\xi))^{1/2} \Theta_\lambda(\xi)^{1/2}} \\ &= \rho(b(\xi))^{1/2} \Theta_\lambda(\xi)^{1/2}. \end{aligned}$$

Thus

$$\begin{aligned} \int_{\beta_{\mathbf{J}(\lambda)}(F_\epsilon)} |r(\mathbf{e}(b(\xi))) \rho(b(\xi)) \Theta_\lambda(\xi)|^2 d\xi &= \int_{\beta_{\mathbf{J}(\lambda)}(F_\epsilon)} \rho(b(\xi)) \Theta_\lambda(\xi) d\xi \\ &= \int_{\beta_{\mathbf{J}(\lambda)}(F_\epsilon)} \rho(b(\xi)) d(b(\xi)) \\ &= \int_{F_\epsilon} \rho(A) d(A) = (*) \end{aligned}$$

Next $d\mu_M(\mathbf{e}(A)) = \rho(A) dA$ implies that

$$(*) = \int_{F_\epsilon} d\mu_M(\mathbf{e}(A)) = \int_{\Omega_\epsilon} d\mu_M(m).$$

Consequently,

$$\int_{\beta_{\mathbf{J}(\lambda)}(F_\epsilon)} |r(\mathbf{e}(b(\xi))) \rho(b(\xi)) \Theta_\lambda(\xi)|^2 d\xi < \infty.$$

Thus, the function

$$\xi \mapsto r(\mathbf{e}(b(\xi))) \rho(b(\xi)) \Theta_\lambda(\xi) \in L^2(\beta_{\mathbf{J}(\lambda)}(F_\epsilon)).$$

Appealing to Lemma 16, we obtain

$$\sum_{\alpha \in \Gamma_P^\epsilon} |\langle \mathbf{h}, \pi_\lambda(\alpha) \mathbf{f}_{r,\epsilon} \rangle|^2 = \left| \det(\mathcal{L}^\epsilon)^\top \right| \int_{F_\epsilon} |\mathbf{h}(\mathbf{e}(A))|^2 [|r(\mathbf{e}(A))|^2 \rho(A)^2 \Theta_\lambda(\beta_{\mathbf{J}(\lambda)}(A))] dA.$$

Next,

$$|r(\mathbf{e}(A))|^2 \rho(A)^2 \Theta_\lambda(\beta_{\mathbf{J}(\lambda)}(A)) = \frac{\rho(A)^2 \Theta_\lambda(\beta_{\mathbf{J}(\lambda)}(A))}{\rho(A) \Theta_\lambda(\beta_{\mathbf{J}(\lambda)}(A))} = \rho(A).$$

Finally

$$\begin{aligned} \sum_{\alpha \in \Gamma_P^\epsilon} |\langle \mathbf{h}, \pi_\lambda(\alpha) \mathbf{f}_{r,\epsilon} \rangle|^2 &= \left| \det(\mathcal{L}^\epsilon)^\top \right| \int_{F_\epsilon} |\mathbf{h}(\mathbf{e}(A))|^2 \underbrace{|r(\mathbf{e}(A))|^2 \rho(A)^2 \Theta_\lambda(\beta_{\mathbf{J}(\lambda)}(A))}_{=\rho(A)} dA \\ &= \left| \det(\mathcal{L}^\epsilon)^\top \right| \int_{F_\epsilon} |\mathbf{h}(\mathbf{e}(A))|^2 \rho(A) dA \\ &= \left| \det(\mathcal{L}^\epsilon)^\top \right| \times \|\mathbf{h}\|_{L^2(\Omega_\epsilon, d\mu_M)}^2. \end{aligned}$$

■

Since $L^2(\Omega_\epsilon, d\mu_M) \cap C(\Omega_\epsilon)$ is dense in $L^2(\Omega_\epsilon, d\mu_M)$ it follows that if $\mathbf{z} \in L^2(\Omega_\epsilon, d\mu_M)$ and

$$r(\mathbf{e}(A)) = \frac{1}{\sqrt{\rho(A) \Theta_\lambda(\beta_{\mathbf{J}(\lambda)}(A))}}$$

then

$$\sum_{\alpha \in \Gamma_P^\epsilon} \left| \langle \mathbf{z}, \pi_\lambda(\alpha) \mathbf{f}_{r,\epsilon} \rangle_{L^2(\Omega_\epsilon, d\mu_M)} \right|^2 = \left| \det(\mathcal{L}^\epsilon)^\top \right| \times \|\mathbf{z}\|_{L^2(\Omega_\epsilon, d\mu_M)}^2.$$

Next, we recall that

$$\{\gamma^{-1}\Omega_\epsilon : \gamma \in \Gamma_M^\epsilon = \exp(\epsilon\mathbb{Z}A_1) \exp(\epsilon\mathbb{Z}A_2) \cdots \exp(\epsilon\mathbb{Z}A_{n_2})\}$$

is a partition of M . Finally, put

$$\Gamma^\epsilon = [\Gamma_M^\epsilon]^{-1} \Gamma_P^\epsilon.$$

Given r such that

$$r(\mathbf{e}(A)) = \frac{1}{\sqrt{\rho(A) \Theta_\lambda(\beta_{\mathbf{J}(\lambda)}(A))}}$$

it follows that

$$\{\pi_\lambda(\kappa)(r \times \mathbf{1}_{\Omega_\epsilon}) : \kappa \in \Gamma^\epsilon\}$$

is a tight frame for $L^2(M, \mu_M)$ with frame bound $|\det(\mathcal{L}^\epsilon)^\top|$.

4.2 Proof of Theorem 3

Fix $\epsilon \in \mathbf{L}$ and define

$$\Omega_\epsilon^\circ = \exp\left(\left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) A_1\right) \cdots \exp\left(\left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) A_{n_2}\right) \subset M$$

such that

$$\mathcal{O}_\epsilon = \left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) A_1 + \cdots + \left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) A_{n_2} \subset \mathfrak{m}$$

is an open set around the neutral element in \mathfrak{m} . By assumption, the restriction of $\beta_{\mathbf{J}(\lambda)}$ to \mathcal{O}_ϵ defines a diffeomorphism between \mathcal{O}_ϵ and $\beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon)$. Next, we define the map

$$\Phi_{\mathbf{J}(\lambda)}^\epsilon : \Omega_\epsilon^\circ \rightarrow \beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon)$$

such that

$$\Phi_{\mathbf{J}(\lambda)}^\epsilon \left(\mathbf{e} \left(\sum_{k=1}^{n_2} a_k A_k \right) \right) = \beta_{\mathbf{J}(\lambda)} \left(\sum_{k=1}^{n_2} a_k A_k \right).$$

Thus,

$$\Phi_{\mathbf{J}(\lambda)}^\epsilon \circ \mathbf{e} = \beta_{\mathbf{J}(\lambda)}$$

is a diffeomorphism (it is a composition of two diffeomorphisms: $\beta_{\mathbf{J}(\lambda)}$ and \mathbf{e}^{-1} .) Let $\mathbf{s} \in C_c^\infty(M)$ such that the support $\text{Supp}(\mathbf{s})$ of \mathbf{s} is a compact subset of Ω_ϵ° . Put

$$\Sigma_{\mathbf{s}} = \Phi_{\mathbf{J}(\lambda)}^\epsilon(\text{Supp}(\mathbf{s})) \subset \beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon).$$

Since $\Phi_{\mathbf{J}(\lambda)}^\epsilon$ is a continuous function, it is clear that $\Sigma_{\mathbf{s}}$ is a compact subset of $\beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon)$.

Lemma 20 *There exists an invertible linear operator $\mathcal{L}^\epsilon : \sum_{j \in \mathbf{J}(\lambda)} \mathbb{R}X_j \rightarrow \sum_{j \in \mathbf{J}(\lambda)} \mathbb{R}X_j$ such that $\Sigma_{\mathbf{s}}$ is contained in a fundamental domain of the lattice*

$$\Lambda_\epsilon^* = \sum_{j \in \mathbf{J}(\lambda)} \mathbb{Z}(\mathcal{L}^\epsilon)^\top X_j^* \subset \mathfrak{p}^*.$$

Proof. Since $\Sigma_{\mathbf{s}}$ is a compact subset of $\sum_{j \in \mathbf{J}(\lambda)} \mathbb{R}X_j$ of positive Lebesgue measure, there exists a positive real number δ_ϵ such that

$$\Sigma_{\mathbf{s}} \subset \sum_{j \in \mathbf{J}(\lambda)} \left[-\frac{\delta_\epsilon}{2}, \frac{\delta_\epsilon}{2} \right) X_j^*.$$

Next, the desired result holds by letting Λ_ϵ^* be equal to $\sum_{j \in \mathbf{J}(\lambda)} \mathbb{Z}\delta_\epsilon X_j^*$. ■

Fix \mathcal{L}^ϵ such that $\Sigma_{\mathbf{s}}$ is contained in a fundamental domain of the lattice

$$\Lambda_\epsilon^* = \sum_{j \in \mathbf{J}(\lambda)} \mathbb{Z}(\mathcal{L}^\epsilon)^\top X_j^* \subset \mathfrak{p}^*.$$

Put $\Lambda_\epsilon = \sum_{j \in \mathbf{J}(\lambda)} \mathbb{Z}\mathcal{L}^\epsilon X_j$ and let $\Gamma_P^\epsilon = \exp(\Lambda_\epsilon) \subset P$. Recall that

$$w(A) = \left| \det \left(\frac{id - e^{-\text{ad}(A)}}{\text{ad}(A)} \right) \right|.$$

Put

$$\mathbf{t}(\xi) = \nu \left(\mathbf{e}^{-1} \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) \right) = (\mathbf{t}_1(\xi), \dots, \mathbf{t}_{n_2}(\xi))$$

and

$$\mathbf{d}(\xi) = \left| \det \begin{bmatrix} \frac{\partial(\mathbf{t}_1(\xi))}{\partial \xi_{j_1}} & \dots & \frac{\partial(\mathbf{t}_1(\xi))}{\partial \xi_{j_{n_2}}} \\ \vdots & \ddots & \vdots \\ \frac{\partial(\mathbf{t}_{n_2}(\xi))}{\partial \xi_{j_1}} & \dots & \frac{\partial(\mathbf{t}_{n_2}(\xi))}{\partial \xi_{j_{n_2}}} \end{bmatrix} \right|. \quad (4.3)$$

Lemma 21 *If $\Psi_{\mathbf{J}(\lambda)}^\epsilon : \beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon) \rightarrow (0, \infty)$ is a positive function given by $\Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi) d\xi = d\mu_M \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right)$ then*

$$\Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi) = w \left(\nu \left(\mathbf{e}^{-1} \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) \right) \right) \mathbf{d}(\xi).$$

Proof. For a suitable positive function \mathbf{f} such that $\mathbf{f} \circ \Phi_{\mathbf{J}(\lambda)}^\epsilon \in L^1(\Omega_\epsilon^\circ, d\mu_M(m))$,

$$\begin{aligned}
& \int_{\Omega_\epsilon^\circ} \mathbf{f} \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(m) \right) d\mu_M(m) \\
&= \int_{\mathcal{O}_\epsilon} \mathbf{f} \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(\mathbf{e}(A)) \right) d\mu_M(\mathbf{e}(A)) \\
&= \int_{\mathcal{O}_\epsilon} \mathbf{f} \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(\exp(\nu(A))) \right) d\mu_M(\exp(\nu(A))) \\
&= \int_{\nu(\mathcal{O}_\epsilon)} \mathbf{f} \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(\exp A) \right) d\mu_M(\exp(A)) \\
&= \int_{\nu(\mathcal{O}_\epsilon)} \mathbf{f} \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(\exp A) \right) \left| \det \left(\frac{id - e^{-\text{ad}(A)}}{\text{ad}(A)} \right) \right| dA.
\end{aligned}$$

The first and second equality above follows from the change of variables $m = \mathbf{e}(A)$, and $\mathbf{e}(A) = \exp(\nu(A))$ respectively. Next,

$$\begin{aligned}
& \int_{\Omega_\epsilon^\circ} \mathbf{f} \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(m) \right) d\mu_M(m) \\
&= \int_{\nu(\mathcal{O}_\epsilon)} \mathbf{f} \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(\exp A) \right) w(A) dA \\
&= \int_{\mathcal{O}_\epsilon} \mathbf{f} \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(\exp(\nu(A))) \right) w(\nu(A)) d(\nu(A)) \\
&= \int_{\mathcal{O}_\epsilon} \mathbf{f} \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(\mathbf{e}(A)) \right) w(\nu(A)) d(\nu(A)) \\
&= \int_{\Omega_\epsilon^\circ} \mathbf{f} \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(m) \right) w(\nu(\mathbf{e}^{-1}(m))) d(\nu(\mathbf{e}^{-1}(m))).
\end{aligned}$$

Finally, the change of variable $\xi = \Phi_{\mathbf{J}(\lambda)}^\epsilon(m)$ yields

$$\begin{aligned}
& \int_{\Omega_\epsilon^\circ} \mathbf{f} \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(m) \right) d\mu_M(m) \\
&= \int_{\beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon)} \mathbf{f}(\xi) w \left(\nu \left(\mathbf{e}^{-1} \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) \right) \right) d \left(\nu \left(\mathbf{e}^{-1} \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) \right) \right) \\
&= \int_{\beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon)} \mathbf{f}(\xi) \underbrace{w \left(\nu \left(\mathbf{e}^{-1} \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) \right) \right) d \left(\nu \left(\mathbf{e}^{-1} \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) \right) \right)}_{=\Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi) d\xi}
\end{aligned}$$

From the definition of \mathbf{d} , it follows that

$$\Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi) = w \left(\nu \left(\mathbf{e}^{-1} \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) \right) \right) \mathbf{d}(\xi).$$

■

Proposition 22 *Let \mathfrak{k}^* be a measurable subset of $\beta_{J(\lambda)}(\mathcal{O}_\epsilon)$. Then*

$$1_{\mathfrak{k}^*} \in L^1(\mathcal{O}_\epsilon, \Psi_{J(\lambda)}^\epsilon(\xi) d\xi) \Leftrightarrow 1_{(\Phi_{J(\lambda)}^\epsilon)^{-1}(\mathfrak{k}^*)} \in L^1(\Omega_\epsilon^\circ, d\mu_M)$$

Proof. First,

$$\int_{\mathfrak{k}^*} \Psi_{J(\lambda)}^\epsilon(\xi) d\xi = \int_{\mathfrak{k}^*} d\mu_M \left((\Phi_{J(\lambda)}^\epsilon)^{-1}(\xi) \right).$$

Second, the change of variable $m = (\Phi_{J(\lambda)}^\epsilon)^{-1}(\xi)$ yields

$$\begin{aligned} \|1_{\mathfrak{k}^*}\|_{L^1(\mathcal{O}_\epsilon, \Psi_{J(\lambda)}^\epsilon(\xi) d\xi)} &= \int_{\mathfrak{k}^*} \Psi_{J(\lambda)}^\epsilon(\xi) d\xi \\ &= \int_{(\Phi_{J(\lambda)}^\epsilon)^{-1}(\mathfrak{k}^*)} d\mu_M(m) \\ &= \left\| 1_{(\Phi_{J(\lambda)}^\epsilon)^{-1}(\mathfrak{k}^*)} \right\|_{L^1(\Omega_\epsilon^\circ, d\mu_M)}. \end{aligned}$$

■

Example 23 *Let $G = \mathbb{R} \rtimes e^\mathbb{R}$ be a simply connected, connected completely solvable Lie group with multiplication law*

$$(x, e^t)(y, e^s) = (x + e^t y, e^{t+s}).$$

Given a linear functional $\lambda = X_1^$, we define the unitary representation π_λ as acting on $L^2((0, \infty), \frac{dh}{h})$ such that*

$$[\pi_\lambda(x, e^t) \mathbf{f}](h) = e^{\frac{2\pi i x}{h}} \mathbf{f}\left(\frac{h}{e^t}\right).$$

Fix $\epsilon = 2 \ln 2$ and define

$$\Omega_{2 \ln 2}^\circ = (e^{-\ln 2}, e^{\ln 2}) = \left(\frac{1}{2}, 2\right)$$

and $\Phi_{J(\lambda)}^{2 \ln 2}(h) = h^{-1}$. Next, let $\mathbf{s} \in C_c^\infty((0, \infty))$ such that the support of \mathbf{s} is a compact subset of $(\frac{1}{2}, 2)$. Then

$$\Sigma_{\mathbf{s}} = \Phi_{J(\lambda)}^\epsilon(\text{Supp}(\mathbf{s})) \subset \left(\frac{1}{2}, 2\right)$$

and $\Sigma_{\mathbf{s}}$ is contained in a fundamental domain of $\Lambda_{2 \ln 2}^ = \frac{3}{2}\mathbb{Z}$. Next, for a suitable function \mathbf{f} ,*

$$\int_{\Omega_\epsilon^\circ} \mathbf{f}(\Phi_{J(\lambda)}^\epsilon(m)) d\mu_M(m) = \int_{(\frac{1}{2}, 2)} \mathbf{f}\left(\frac{1}{h}\right) \frac{dh}{h} = \int_{(\frac{1}{2}, 2)} \mathbf{f}(\xi) \frac{d\xi}{\xi}.$$

Thus

$$\Psi_{J(\lambda)}^{2 \ln 2}(\xi) = \xi^{-1}$$

and

$$\Psi_{J(\lambda)}^{2 \ln 2}(\Phi_{J(\lambda)}^{2 \ln 2}(h)) = \Psi_{J(\lambda)}^{2 \ln 2}(h^{-1}) = h.$$

Proposition 24 *If \mathbf{K} is a compact subset of Ω_ϵ° then*

$$m \mapsto \left[\Psi_{\mathbf{J}(\lambda)}^\epsilon \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(m) \right) \right]^{-1} \in L^1(\mathbf{K}, d\mu_M).$$

Proof. Let \mathbf{K} be a compact subset of Ω_ϵ° . In order to prove this result, it suffices to establish that

$$\int_{\mathbf{K}} \frac{d\mu_M(m)}{\Psi_{\mathbf{J}(\lambda)}^\epsilon \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(m) \right)} = \int_{\Phi_{\mathbf{J}(\lambda)}^\epsilon(\mathbf{K})} d\xi.$$

The change of variable $\xi = \Phi_{\mathbf{J}(\lambda)}^\epsilon(m)$ yields

$$\begin{aligned} \int_{\mathbf{K}} \frac{d\mu_M(m)}{\Psi_{\mathbf{J}(\lambda)}^\epsilon \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(m) \right)} &= \int_{\Phi_{\mathbf{J}(\lambda)}^\epsilon(\mathbf{K})} \frac{d\mu_M \left(\left(\Phi_{\mathbf{J}(\lambda)}^\epsilon \right)^{-1}(\xi) \right)}{\Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi)} \\ &= \int_{\Phi_{\mathbf{J}(\lambda)}^\epsilon(\mathbf{K})} \frac{1}{\Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi)} \Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi) d\xi \\ &= \int_{\Phi_{\mathbf{J}(\lambda)}^\epsilon(\mathbf{K})} d\xi. \end{aligned}$$

Since $\Phi_{\mathbf{J}(\lambda)}^\epsilon$ is continuous and \mathbf{K} is compact, we obtain

$$\int_{\mathbf{K}} \frac{d\mu_M(m)}{\Psi_{\mathbf{J}(\lambda)}^\epsilon \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(m) \right)} = \left| \Phi_{\mathbf{J}(\lambda)}^\epsilon(\mathbf{K}) \right| < \infty.$$

■

Lemma 25 *Let $\mathbf{s} \in C_c^\infty(M)$ such that the support $\text{Supp}(\mathbf{s})$ of \mathbf{s} is a compact subset of Ω_ϵ° . Then \mathbf{s} is square-integrable with respect to the Haar measure $d\mu_M(m)$.*

Proof. Recall that $\nu, \mathbf{e}, \Phi_{\mathbf{J}(\lambda)}^\epsilon$ are smooth bijective functions and

$$\Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi) = w \left(\nu \left(\mathbf{e}^{-1} \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) \right) \right) \mathbf{d}(\xi).$$

As such, $\Psi_{\mathbf{J}(\lambda)}^\epsilon$ is a positive smooth function defined on

$$\beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon) \supset \Sigma_{\mathbf{s}}.$$

Next,

$$\begin{aligned}
\int_M |\mathbf{s}(m)|^2 d\mu_M(m) &= \int_{\Sigma_{\mathbf{s}}} \left| \mathbf{s} \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) \right|^2 d\mu_M \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) \\
&= \int_{\Sigma_{\mathbf{s}}} \left| \mathbf{s} \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) \right|^2 \Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi) d\xi \\
&= \int_{\Sigma_{\mathbf{s}}} \left| \mathbf{s} \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) (\Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi))^{1/2} \right|^2 d\xi \\
&\leq \int_{\Sigma_{\mathbf{s}}} \left\| \xi \mapsto \mathbf{s} \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) (\Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi))^{1/2} \right\|_{L^\infty(\Sigma_{\mathbf{s}})}^2 d\xi \\
&= \left\| \xi \mapsto \mathbf{s} \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) (\Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi))^{1/2} \right\|_{L^\infty(\Sigma_{\mathbf{s}})}^2 |\Sigma_{\mathbf{s}}|.
\end{aligned}$$

The compactness of $\Sigma_{\mathbf{s}}$ together with the fact that the function

$$\xi \mapsto \mathbf{s} \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) (\Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi))^{1/2}$$

is uniformly continuous on $\Sigma_{\mathbf{s}}$ gives the desired result. ■

Lemma 26 *Let \mathbf{s}_1 be defined such that*

$$\mathbf{s}_1(m) = \mathbf{s}(m) \sqrt{\Psi_{\mathbf{J}(\lambda)}^\epsilon \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(m) \right)}. \quad (4.4)$$

Then \mathbf{s}_1 is a smooth function of compact support.

Proof. Since

$$\Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi) = w \left(\nu \left(\mathbf{e}^{-1} \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) \right) \right)$$

it is clear that

$$\Psi_{\mathbf{J}(\lambda)}^\epsilon : \beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon) \rightarrow (0, \infty)$$

is a smooth positive function defined on $\beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon)$. Next, since \mathbf{s}_1 is the product of the smooth functions $\sqrt{\Psi_{\mathbf{J}(\lambda)}^\epsilon \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(m) \right)}$ and $\mathbf{s}(m)$, and since $\sqrt{\Psi_{\mathbf{J}(\lambda)}^\epsilon \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(m) \right)}$ is a positive function, it follows that \mathbf{s}_1 is a smooth function which is supported on $\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\Sigma_{\mathbf{s}})$. ■

Next, let Γ_M be a discrete subset of M . Next, let \mathbf{h} be a continuous function which is also square-integrable with respect to the Haar measure. Then

$$\begin{aligned}
&\sum_{\exp(X) \in \Gamma_P^\epsilon} \sum_{\gamma \in \Gamma_M} |\langle \mathbf{h}, \pi_\lambda(\gamma^{-1}) \pi_\lambda(\exp X) \mathbf{s} \rangle|^2 \\
&= \sum_{\exp(X) \in \Gamma_P^\epsilon} \sum_{\gamma \in \Gamma_M} \left| \int_M \mathbf{h}(m) e^{-2\pi i \langle ([Ad(\gamma m)]^{-1})^* \lambda, X \rangle} \overline{\mathbf{s}(\gamma m)} d\mu_M(m) \right|^2.
\end{aligned}$$

The change of variable $n = \gamma m$ yields

$$\begin{aligned} & \sum_{\exp(X) \in \Gamma_P^\epsilon} \sum_{\gamma \in \Gamma_M} \left| \int_M \mathbf{h}(m) e^{-2\pi i \langle ([Ad(\gamma m)]^{-1})^* \lambda, X \rangle} \overline{\mathbf{s}(\gamma m)} d\mu_M(m) \right|^2 \\ &= \sum_{\exp(X) \in \Gamma_P^\epsilon} \sum_{\gamma \in \Gamma_M} \left| \int_M \mathbf{h}(\gamma^{-1}n) e^{-2\pi i \langle ([Ad(n)]^{-1})^* \lambda, X \rangle} \overline{\mathbf{s}(n)} d\mu_M(\gamma^{-1}n) \right|^2. \end{aligned}$$

Secondly, since

$$\langle ([Ad(n)]^{-1})^* \lambda, X \rangle = \langle \Phi_{\mathbf{J}(\lambda)}^\epsilon(n), X \rangle$$

it follows that

$$\begin{aligned} & \sum_{\gamma \in \Gamma_M} \sum_{\exp(X) \in \Gamma_P^\epsilon} |\langle \mathbf{h}, \pi_\lambda(\gamma^{-1}) \pi_\lambda(\exp X) \mathbf{s} \rangle|^2 \\ &= \sum_{\gamma \in \Gamma_M} \sum_{\exp(X) \in \Gamma_P^\epsilon} \left| \int_M \mathbf{h}(\gamma^{-1}n) e^{-2\pi i \langle \Phi_{\mathbf{J}(\lambda)}^\epsilon(n), X \rangle} \overline{\mathbf{s}(n)} d\mu_M(\gamma^{-1}n) \right|^2 \\ &= \sum_{\gamma \in \Gamma_M} \sum_{\exp(X) \in \Gamma_P^\epsilon} \left| \int_M \mathbf{h}(\gamma^{-1}n) e^{-2\pi i \langle \Phi_{\mathbf{J}(\lambda)}^\epsilon(n), X \rangle} \overline{\mathbf{s}(n)} d\mu_M(n) \right|^2 \\ &= \sum_{\gamma \in \Gamma_M} \sum_{\exp(X) \in \Gamma_P^\epsilon} \left| \int_{\Omega_\epsilon^\circ} \mathbf{h}(\gamma^{-1}n) e^{-2\pi i \langle \Phi_{\mathbf{J}(\lambda)}^\epsilon(n), X \rangle} \overline{\mathbf{s}(n)} d\mu_M(n) \right|^2 = (*) \end{aligned}$$

Thirdly, let $\xi = \Phi_{\mathbf{J}(\lambda)}^\epsilon(n)$ and define

$$\mathbf{z}_\gamma(\xi) = \mathbf{h} \left(\gamma^{-1} \left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) \overline{\mathbf{s} \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right)}.$$

Since

$$d\mu_M \left(\left(\Phi_{\mathbf{J}(\lambda)}^\epsilon \right)^{-1}(\xi) \right) = \Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi) d\xi$$

it follows that

$$\begin{aligned} & \int_{\Omega_\epsilon^\circ} \mathbf{h}(\gamma^{-1}n) e^{-2\pi i \langle \Phi_{\mathbf{J}(\lambda)}^\epsilon(n), X \rangle} \overline{\mathbf{s}(n)} d\mu_M(n) \\ &= \int_{\beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon)} \mathbf{h} \left(\gamma^{-1} \left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) e^{-2\pi i \langle \xi, X \rangle} \overline{\mathbf{s} \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right)} d\mu_M \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) \\ &= \int_{\beta_{\mathbf{J}(\lambda)}(\mathcal{O}_\epsilon)} \underbrace{\left(\mathbf{h} \left(\gamma^{-1} \left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) \overline{\mathbf{s} \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right)} \right)}_{=\mathbf{z}_\gamma(\xi)} \underbrace{e^{-2\pi i \langle \xi, X \rangle} d\mu_M \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right)}_{=\Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi) d\xi}. \end{aligned}$$

As a consequence of the observations made above

$$\begin{aligned}
(*) &= \sum_{\gamma \in \Gamma_M} \sum_{\exp(X) \in \Gamma_P^\epsilon} \left| \int_{\beta_{J(\lambda)}(\mathcal{O}_\epsilon)} \mathbf{z}_\gamma(\xi) \times e^{-2\pi i \langle \xi, X \rangle} \times \Psi_{J(\lambda)}^\epsilon(\xi) d\xi \right|^2 \\
&= \sum_{\gamma \in \Gamma_M} \sum_{\exp(X) \in \Gamma_P^\epsilon} \left| \int_{\beta_{J(\lambda)}(\mathcal{O}_\epsilon)} [\mathbf{z}_\gamma(\xi) \Psi_{J(\lambda)}^\epsilon(\xi)] e^{-2\pi i \langle \xi, X \rangle} d\xi \right|^2 \\
&= (**)
\end{aligned}$$

Next, note that the support of the function

$$\xi \mapsto \mathbf{s} \left(\left[\Phi_{J(\lambda)}^\epsilon \right]^{-1}(\xi) \right)$$

is a compact subset $\Sigma_{\mathbf{s}}$ of $\beta_{J(\lambda)}(\mathcal{O}_\epsilon)$ which is contained in a fundamental domain of Λ_ϵ^* . Moreover, the trigonometric system

$$\left\{ \xi \mapsto \frac{e^{-2\pi i \langle \xi, X \rangle}}{|\det(\mathcal{L}^\epsilon)^\top|^{1/2}} : X \in \Lambda_\epsilon \right\}$$

is an orthonormal basis for $L^2(\mathfrak{p}^*/\Lambda_\epsilon^*)$. Next,

$$\begin{aligned}
(**) &= \sum_{\gamma \in \Gamma_M} \sum_{\exp(X) \in \Gamma_P^\epsilon} \left| \int_{\Sigma_{\mathbf{s}}} \left[|\det(\mathcal{L}^\epsilon)^\top|^{1/2} \mathbf{z}_\gamma(\xi) \Psi_{J(\lambda)}^\epsilon(\xi) \right] \frac{e^{-2\pi i \langle \xi, X \rangle}}{|\det(\mathcal{L}^\epsilon)^\top|^{1/2}} d\xi \right|^2 \\
&= \sum_{\gamma \in \Gamma_M} \int_{\Sigma_{\mathbf{s}}} \left| |\det(\mathcal{L}^\epsilon)^\top|^{1/2} \mathbf{z}_\gamma(\xi) \Psi_{J(\lambda)}^\epsilon(\xi) \right|^2 d\xi \\
&= \sum_{\gamma \in \Gamma_M} \int_{\Sigma_{\mathbf{s}}} \left| |\det(\mathcal{L}^\epsilon)^\top|^{1/2} \mathbf{z}_\gamma(\xi) \Psi_{J(\lambda)}^\epsilon(\xi)^{1/2} \right|^2 \Psi_{J(\lambda)}^\epsilon(\xi) d\xi \\
&= (***)
\end{aligned}$$

Setting

$$\left[\Phi_{J(\lambda)}^\epsilon \right]^{-1}(\xi) = n$$

yields

$$\begin{aligned}
\mathbf{z}_\gamma(\xi) &= \mathbf{h} \left(\gamma^{-1} \left[\Phi_{J(\lambda)}^\epsilon \right]^{-1}(\xi) \right) \overline{\mathbf{s} \left(\left[\Phi_{J(\lambda)}^\epsilon \right]^{-1}(\xi) \right)} \\
&= \mathbf{h} \left(\gamma^{-1} \left[\Phi_{J(\lambda)}^\epsilon \right]^{-1} \left(\Phi_{J(\lambda)}^\epsilon(n) \right) \right) \overline{\mathbf{s} \left(\left[\Phi_{J(\lambda)}^\epsilon \right]^{-1} \left(\Phi_{J(\lambda)}^\epsilon(n) \right) \right)} \\
&= \mathbf{h}(\gamma^{-1}n) \overline{\mathbf{s}(n)}
\end{aligned}$$

and

$$\begin{aligned}
(***) &= \sum_{\gamma \in \Gamma_M^\epsilon} \int_{[\Phi_{\mathbf{J}(\lambda)}^\epsilon]^{-1}(\Sigma_{\mathbf{s}})} \left| \left| \det(\mathcal{L}^\epsilon)^\top \right|^{1/2} \mathbf{h}(\gamma^{-1}n) \overline{\mathbf{s}(n)} \left[\Psi_{\mathbf{J}(\lambda)}^\epsilon \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(n) \right) \right]^{1/2} \right|^2 \\
&\quad \times \left(\Psi_{\mathbf{J}(\lambda)}^\epsilon \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(n) \right) \right) d\left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(n) \right).
\end{aligned}$$

Since

$$d\mu_M \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1}(\xi) \right) = \Psi_{\mathbf{J}(\lambda)}^\epsilon(\xi) d\xi$$

it follows that

$$\begin{aligned}
&\sum_{\gamma \in \Gamma_M} \int_{[\Phi_{\mathbf{J}(\lambda)}^\epsilon]^{-1}(\Sigma_{\mathbf{s}})} \left| \left| \det(\mathcal{L}^\epsilon)^\top \right|^{1/2} \mathbf{h}(\gamma^{-1}n) \overline{\mathbf{s}(n)} \left[\Psi_{\mathbf{J}(\lambda)}^\epsilon \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(n) \right) \right]^{1/2} \right|^2 \\
&\quad \times \Psi_{\mathbf{J}(\lambda)}^\epsilon \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(n) \right) d\left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(n) \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{\exp(X) \in \Gamma_P^\epsilon} \sum_{\gamma \in \Gamma_M} \left| \langle \mathbf{h}, \pi_\lambda(\gamma^{-1}) \pi_\lambda(\exp X) \mathbf{s} \rangle \right|^2 \\
&= \sum_{\gamma \in \Gamma_M} \int_{[\Phi_{\mathbf{J}(\lambda)}^\epsilon]^{-1}(\Sigma_{\mathbf{s}})} \left| \left| \det(\mathcal{L}^\epsilon)^\top \right|^{1/2} \mathbf{h}(\gamma^{-1}n) \overline{\mathbf{s}(n)} \left[\Psi_{\mathbf{J}(\lambda)}^\epsilon \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(n) \right) \right]^{1/2} \right|^2 \\
&\quad \times d\mu_M \left(\left[\Phi_{\mathbf{J}(\lambda)}^\epsilon \right]^{-1} \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(n) \right) \right) \\
&= \sum_{\gamma \in \Gamma_M} \int_{[\Phi_{\mathbf{J}(\lambda)}^\epsilon]^{-1}(\Sigma_{\mathbf{s}})} \left| \left| \det(\mathcal{L}^\epsilon)^\top \right|^{1/2} \mathbf{h}(\gamma^{-1}n) \overline{\mathbf{s}(n)} \left[\Psi_{\mathbf{J}(\lambda)}^\epsilon \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(n) \right) \right]^{1/2} \right|^2 d\mu_M(n) \\
&= \sum_{\gamma \in \Gamma_M} \int_{[\Phi_{\mathbf{J}(\lambda)}^\epsilon]^{-1}(\Sigma_{\mathbf{s}})} |\mathbf{h}(\gamma^{-1}n)|^2 \left| \left| \det(\mathcal{L}^\epsilon)^\top \right|^{1/2} \mathbf{s}(n) \left[\Psi_{\mathbf{J}(\lambda)}^\epsilon \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(n) \right) \right]^{1/2} \right|^2 d\mu_M(n) \\
&= (***)
\end{aligned}$$

The change of variable $m = \gamma^{-1}n$ yields

$$\begin{aligned}
&|\mathbf{h}(\gamma^{-1}n)|^2 \left| \left| \det(\mathcal{L}^\epsilon)^\top \right|^{1/2} \mathbf{s}(n) \left[\Psi_{\mathbf{J}(\lambda)}^\epsilon \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(n) \right) \right]^{1/2} \right|^2 \\
&= |\mathbf{h}(m)|^2 \left| \left| \det(\mathcal{L}^\epsilon)^\top \right|^{1/2} \mathbf{s}(\gamma m) \left[\Psi_{\mathbf{J}(\lambda)}^\epsilon \left(\Phi_{\mathbf{J}(\lambda)}^\epsilon(\gamma m) \right) \right]^{1/2} \right|^2.
\end{aligned}$$

Since $d\mu_M$ is a Haar measure, then

$$\begin{aligned}
& (* **) \\
& = \sum_{\gamma \in \Gamma_M} \int_{\gamma^{-1} [\Phi_{J(\lambda)}^\epsilon]^{-1}(\Sigma_{\mathbf{s}})} |\mathbf{h}(m)|^2 \left| \left| \det(\mathcal{L}^\epsilon)^\top \right|^{1/2} \mathbf{s}(\gamma m) \left[\Psi_{J(\lambda)}^\epsilon \left(\Phi_{J(\lambda)}^\epsilon(\gamma m) \right) \right]^{1/2} \right|^2 d\mu_M(m) \\
& = \sum_{\gamma \in \Gamma_M} \int_{\gamma^{-1} [\Phi_{J(\lambda)}^\epsilon]^{-1}(\Sigma_{\mathbf{s}})} |\mathbf{h}(m)|^2 \left| \sqrt{\Psi_{J(\lambda)}^\epsilon \left(\Phi_{J(\lambda)}^\epsilon(\gamma m) \right)} \left| \det(\mathcal{L}^\epsilon)^\top \right| \mathbf{s}(\gamma m) \right|^2 d\mu_M(m) \\
& = \int_M |\mathbf{h}(m)|^2 \left(\sum_{\gamma \in \Gamma_M} \left| \sqrt{\Psi_{J(\lambda)}^\epsilon \left(\Phi_{J(\lambda)}^\epsilon(\gamma m) \right)} \left| \det(\mathcal{L}^\epsilon)^\top \right| \mathbf{s}(\gamma m) \right|^2 \right) d\mu_M(m).
\end{aligned}$$

Next, let

$$\mathcal{Z}^\epsilon = \left\{ \Gamma \text{ is a discrete subset of } M \mid \inf_{m \in M} \left(\sum_{\gamma \in \Gamma} \left| \sqrt{\Psi_{J(\lambda)}^\epsilon \left(\Phi_{J(\lambda)}^\epsilon(\gamma m) \right)} \left| \det(\mathcal{L}^\epsilon)^\top \right| \mathbf{s}(\gamma m) \right|^2 \right) > 0 \right. \\
\left. \sup_{m \in M} \left(\sum_{\gamma \in \Gamma_M} \left| \sqrt{\Psi_{J(\lambda)}^\epsilon \left(\Phi_{J(\lambda)}^\epsilon(\gamma m) \right)} \left| \det(\mathcal{L}^\epsilon)^\top \right| \mathbf{s}(\gamma m) \right|^2 \right) < \infty \right\}.$$

Since

$$\mathbf{s}(m) \sqrt{\Psi_{J(\lambda)}^\epsilon \left(\Phi_{J(\lambda)}^\epsilon(m) \right)}$$

is a smooth function of compact support, \mathcal{Z}^ϵ is non-empty. Fixing $\Gamma_M^\epsilon \in \mathcal{Z}^\epsilon$, it is clear that there exists $A_{\mathbf{s}, \Gamma_M^\epsilon, \Lambda^*} > 0$, and $B_{\mathbf{s}, \Gamma_M^\epsilon, \Lambda^*} < \infty$ such that

$$\begin{aligned}
A_{\mathbf{s}, \Gamma_M^\epsilon, \Lambda^*} \times \|\mathbf{h}\|_{L^2(M, d\mu_M)}^2 & \leq \sum_{\exp(X) \in \Gamma_P^\epsilon} \sum_{\gamma \in \Gamma_M^\epsilon} |\langle \mathbf{h}, \pi_\lambda(\gamma^{-1}) \pi_\lambda(\exp X) \mathbf{s} \rangle|^2 \\
& \leq B_{\mathbf{s}, \Gamma_M^\epsilon, \Lambda^*} \times \|\mathbf{h}\|_{L^2(M, d\mu_M)}^2
\end{aligned}$$

and

$$\{ \pi_\lambda(\kappa) \mathbf{s} : \kappa \in (\Gamma_M^\epsilon)^{-1} \Gamma_P^\epsilon \}$$

is a frame generated by a smooth function of compact support.

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